BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES
MATHÉMATIQUES, ASTRONOMIQUES ET PHYSIQUES

Volume IX Numéro 12

VARSOVIE 1961

Le Bulletin de l'Académie Polonaise des Sciences est publié en Séries.

La présente Série comprend des communications du domaine de la mathématique, astronomie et physique

Lesdites sciences sont du domaine de la Troisième Classe de l'Académie Polonaise des Sciences.

La présente Série paraît mensuellement.

Les communications destinées à être publiées dans le Bulletin sont présentées à la Rédaction du Bulletin par les membres de l'Académie.

Adresse de la Rédaction: Palac Kultury i Nauki, Varsovie

BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

Rédacteur en chef
K. KURATOWSKI

Rédacteur en chef suppléant
L. INFELD

SÉRIE DES SCIENCES
MATHÉMATIQUES, ASTRONOMIQUES ET PHYSIQUES

Rédacteur de la Série
A. MOSTOWSKI

Comité de Rédaction de la Série
K. BORSUK, W. IWANOWSKA, A. JABŁOŃSKI, W. RUBINOWICZ

VOLUME IX
NUMÉRO 12

VARSOVIE 1961

PRINTED IN POLAND

PAŃSTWOWE WYDAWNICTWO NAUKOWE, W-WA, MIODOWA 10

Nakład 1089+161 egz.	*Rękopis dostarczono 4.X.1961*
Ark. wyd. 5,75, druk. 3 75+0,25	*Podpisano do druku 2.XII.61*
Papier druk. sat. 80 g kl. III	*Druk ukończono w grudniu 1961*
Format B5, 70×100 cm	*Zam. prod. 420/61* *Cena zł 20.—*

WARSZAWSKA DRUKARNIA NAUKOWA, W-WA, ŚNIADECKICH 8

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 12, 1961

MATHEMATIK

Distributionen auf Yamabe-Gruppen. Harmonische Analyse auf einer Abelschen l. k. Gruppe

von

K. MAURIN

Vorgelegt von A. MOSTOWSKI am 11 September 1961

In letzter Zeit hat man auf verschiedene Weise Sätze von Plancherel und Bochner verallgemeinert. In [4] hat der Verfasser ein allgemeines Schema entworfen und gezeigt, dass es im Falle der Lieschen — oder diskreten Gruppen die Sätze von Weil, Rajkow, Berežanskij, Källén, Lehmann, Maurin u.a. umfasst.

In dieser Note zeigen wir, dass man aus unseren Sätzen (vgl. [4]) und einem Satze von R. Godement mühelos nicht nur die fundamentalen Sätze der harmonischen Analyse auf Abelschen Gruppen von Weil, Krein und Rajkow sondern sogar eine weitgehende Verallgemeinerung des berühmten Satzes von L. Schwartz über positivdefinite Distributionen erhalten kann *).

Zu diesem Zwecke wird auf einer weiten Klasse lokal kompakter (l.k.) Gruppen ein nuklearer Funktionenraum Φ konstruiert **). Die stetigen linearen Funktionale, auf Φ nennen wir Distributionen auf l.k. Gruppen, da sie eine natürliche Verallgemeinerung der Distributionen auf Lieschen Gruppen sind.

Im Falle der kommutativen l.k.G. zeigen wir, dass die Charaktere Eigendistributionen der Verschiebungsoperatoren sind.

1. Distributionen auf separablen l. k. Gruppen

Es sei G eine separable l.k. Gruppe und dx das linksinvariante Mass auf G. Die (linke) Verschiebungen $x \to sx$ induzieren bekanntlich eine unitäre Darstellung (U_s, G) von G im Raume $L^2(G)$ $\quad (U_s f)(x) \overset{\text{df}}{=} f(sx), \quad f \in L^2(G)$.

DEFINITION. G heisst Yamabe-Gruppe falls in jeder kompakten Umgebung des Einzelelementes e eine solche invariante Untergruppe g_n existiert, dass $G_n = G/g_n$ eine Lie-Gruppe ist.

Im folgenden werden Yamabe- und diskreten (Y.D-) Gruppen untersucht.

*) Es scheint, dass R. Godement in [2] eine ähnliche Verallgemeinerung des Satzes von Schwartz-Bochner im Sinne hatte.

**) Dieser Raum wurde in einer sehr interessanten Note [3] von G. I. Kac erwähnt.

Diese Klasse umfasst alle für Anwendungen wichtigen Fälle. Bekanntlich gehören: a) abelsche, b) kompakte, c) zusamenhängende l.k. Gruppen, d) lokal nilpotente durch eine kompakte Untermenge erzeugte, e) diskrete separate Gruppen zu dieser Klasse.

SATZ 1. *Es sei G eine Yamabe oder diskrete Gruppe und es sei Φ der induktive Limes der Räume Φ_n: $\Phi = \text{l. ind}_n \Phi_n$, wobei im Yamabe-Fall $\Phi_n = D(G/g_n)$ im diskreten dagegen $\Phi_n = C_0(K_n)$ wo K_n kompakte (also endliche) gegen e konvergente Untermengen sind: $K_n \nearrow G$*) d. h. $\Phi = U_n \Phi_n$ wobei Φ mit einer solchen starksten lokalkonvexen Topologie ausgestattet ist, dass die Einbettungen $\pi_n : \Phi \to \Phi_n$ (noch) stetig sind. Dann ist Φ 1° nuklear, 2° gleichmässig dicht in $C_0(G)$, 3° dicht $L^p(G)$, $p \geqslant 1$, 4° Die (linke) Verschiebungsoperatoren U_s induzieren eine starkstetige Darstellung von G in Φ:*

$$G \ni s \to U_s \varphi \in \Phi \quad \textit{ist stetig für jedes } \varphi \in \Phi, \tag{1.1}$$

5° die Operatoren U_s sind stetig (also nuklear) in Φ.

Beweis. Wir untersuchen zuerst den diskreten Fall.

Ad 1°. Es sei also G diskret und $K_1 \subset K_2 \subset \ldots$, K_n-kompakt und $K_n \nearrow G$. $\Phi_n = C_0(K_n)$ ist endlichdimensional, also nuklear, da K_n endlich ist. Da der (strikte) induktive Limes der nuklearen Räume nuklear ist, ist 1° bewiesen. Punkte 2°—5° sind in diesem Fall trivial.

Der Yamabe-Fall. Wir identifizieren — wie üblich — eine Funktion $\dot{\varphi}$ auf der Faktorgruppe G/g_n mit der auf G definierten Funktion φ, die auf Nebenklassen $\dot{x} = x \cdot g_n$ des Normalteilers g_n konstant ist $\dot{\varphi}(\dot{x}) = \varphi(x) = \varphi(y)$ identisch für $y^{-1} x \in g_n$. Also $\Phi_{n+1} = D(G/g_{n+1}) \supset D(G/g_n) = \Phi_n$, da $g_n \supset g_{n+1}$.

Weil $G_n = G/g_n$ eine (separable) Liesche Gruppe ist, sind bekanntlich alle Räume $\Phi_n = D(G_n)$ nuklear. Die Einbettungen $\pi^n_{n+1} : \Phi_n \to \Phi_{n+1}$ sind stetig (weil linear und abgeschlossen), $\Phi = \text{l. ind } \Phi_n$ ist also als induktiver Limes nuklearer Räume nuklear.

Ad 2°. Es sei $f \in C_0(G)$. Da f gleichmässig stetig auf G ist, gibt es eine solche symmetrische Umgebung N_n des Einheitselements e, dass $|f(x) - f(y)| < 1/n$ für alle $x, y \in G$ für welche $x^{-1} y \in N_n$.

Es sei g_n eine solche kompakte invariante Untergruppe in N_n, dass $G_n = G/g_n$ Liesch ist. Es sei $f_n(x) \overset{\text{df}}{=} |g_n|^{-1} \int_{g_n} f(x\xi)\, d\xi$, wo $d\xi$ ein invariantes Mass auf g_n und $|g_n| = \int_{g_n} d\xi$. Dann haben wir für jedes $\sigma \in g_n$

$$f_n(x\sigma) = \frac{1}{|g_n|} \int_{g_n} f(x\sigma\xi)\, d\xi = \frac{1}{|g_n|} \int_{g_n} f(x\sigma\sigma^{-1}\xi)\, d\xi = f_n(x) \tag{1.2}$$

*) Falls G_n eine Liesche Gru pe ist, bedeutet $D(G_n)$ den Raum $C_0^\infty(G_n)$ der beliebig oft differenzierba en Funktionen mit kompakten Trägern, der mit einer nuklearen Topologie von L. Schwartz ausg stattet ist. $C_0(K)$ ist der Raum der stetigen Funktionen mit kompakten Trägern in K.

f_n ist also konstant auf den Nebenklassen von g. Es gilt

$$(2) \quad |f(x)-f_n(x)| = \left| \frac{1}{|g_n|} \int_{g_n} f(x)\, d\xi - f_n(x) \right| \leqslant$$

$$\leqslant \frac{1}{|g_n|} \int_{g_n} |f(x)-f(x\xi)|\, d\xi < \frac{1}{n} \cdot |g_n|^{-1} \int_{g_n} d\xi = \frac{1}{n}.$$

Da man jede stetige Funktion auf einer Lieschen Gruppe G_n gleichmässig durch Funktionen aus $D(G_n)$ approximieren kann, gibt es eine solche Funktion $\varphi_n \in D(G(g_n))$, dass

$$(3) \quad |f_n(x) - \varphi_n(x)| < \frac{1}{n}.$$

Aus (2) und (3) folgt die Behauptung. Ad 3°: Folgt unmittelbar aus 2°. Ad 4°: Bekanntlich ist auf der Lieschen Gruppe $G_n = G/g_n$ die reguläre Darstellung $G_n \ni s \to U_s \varphi_n \in D(G_n)$ stetig für jedes $\varphi_n \in D(G_n)$. Aber, da die Einbettungen $\pi_n : \Phi_n \to \Phi$ stetig sind, ist die Darstellung

$$G \ni s \to U_s \varphi \in \Phi \quad \text{stetig für jedes} \quad \varphi \in \Phi.$$

Ad 5°. Die Einschränkung $U_s|\Phi_n$ ist stetig auf Φ_n weil linear und abgeschlossen. Da $\Phi = \text{l. ind } \Phi_n$, bildet $U_s|\Phi$ stetig Φ in sich ab.

Der Satz ist also vollständig bewiesen.

Definition. Wir werden den im Satze 1 konstruierten nuklearen Raum im folgenden mit $D(G)$ bezeichnen, weil er eine natürliche Verallgemeinerung des Schwartzschen Funktionenraumes im Falle der Lieschen Gruppe ist.

Es sei $\langle \varphi, T \rangle$ die Dualität von Φ und Φ', die

$$(4) \quad \langle \varphi, f \rangle = \int_G \varphi(x) \overline{f(x)}\, dx = (\varphi|f)_{L^2(G)} \quad \text{erweitert.}$$

Dann nennen wir die Elemente von $D'(G) = \Phi'$ „*Distributionen auf der l.k. Gruppe G*" Wir haben also das folgende.

Korollar. *$D(G) \subset C_0(G) \subset L^p(G) \subset D'(G)$, wobei jeder Raum im folgenden dicht ist.*

Bemerkung. Im Falle der diskreten Gruppe G sind Elemente von $D(G)$ Masse auf G.

Aus der Stetigkeit der Faltung auf Lieschen Gruppen folgt

Satz 2. *Die Faltung $S_\psi : \Phi \ni \varphi \to \psi * \varphi \in \Phi$ ist linear und stetig.*

2. Harmonische Analyse auf einer kommutativen l. k. Gruppe

Wir werden in folgendem einen sehr speziellen Fall unseres Spektralsatzes gebrauchen (vgl. [4], [5]), den wir jetzt anführen als den

Satz 3. Voraussetzung: *Es sei Φ ein nuklearer Raum und es sei (U_g) ein vertauschbares System der stetigen Automorphismen von Φ. Es sei ein hermitischer*

positivdefiniter (U_g)-invarianter Kern auf Φ: d. h. die bilineare Form $(\varphi, \psi) \to B(\varphi, \psi)$ ist separat stetig, $B(\varphi, \varphi) \geqslant 0$; $B(U_g \varphi, U_g \psi) = B(\varphi, \psi)$ identisch für $\varphi, \psi \in \Phi$, $U_g \in (U_g)$.

Es sei N der B-Nullunterraum von Φ, $\dot{\Phi} = \Phi/N$, $P = P(B)$ die kanonische Projektion $\Phi \to \Phi/N$. H ist der durch die Vervollständigung von $\dot{\Phi}$ in der Norm $\|\varphi\|^2 = B(\varphi, \varphi)$ erhaltene Hilbertsche Raum. Es sei $\dot{U}_g$ der von U_g induziert unitäre Operator auf H. Es sei die auf $(\dot{U}_g)$ aufgespannte v. Neumannsche Algebra $A = A((\dot{U}_g))$ maximal.

Behauptung. *Dann gibt es einen kompakten Raum Λ und ein positives Mass μ auf Λ and eine solche unitäre Abbildung $F: H \to L^2(\Lambda, \mu)$ von H auf $L^2(\mu)$, dass:*

$$(F\dot{U}_g f)(\lambda) = \dot{U}_g\hat{}\,(\lambda) Ff(\lambda), \quad \text{wobei} \quad \dot{U}_g\hat{}\,(\lambda) \in C^1, \tag{2.1}$$

$$U_g\hat{}\,(\cdot) \quad \text{stetig auf } \Lambda \text{ sind.} \tag{2.2}$$

$$(U_{g_1} \cdot U_{g_2})\hat{}\,(\lambda) = \dot{U}_{g_1}\hat{}\,(\lambda)\, \dot{U}_{g_2}\hat{}\,(\lambda),$$

$$(\dot{U}_{g^{-1}})\hat{}\,(\lambda) = (\dot{U}_g)\hat{}\,(\lambda)^{-1} = \overline{\dot{U}_g\hat{}\,(\lambda)}, \tag{2.3}$$

$$|U_g\hat{}\,(\lambda)| \equiv 1. \tag{2.4}$$

Für jedes $\dot{\varphi} \in \dot{\Phi}$ ist die $(\dot{U}_g)$ — Fouriertransformierte von $\dot{\varphi}\hat{}$ durch funktionale Form

$$\dot{\varphi}\hat{}\,(\lambda) = \langle \varphi, e_\lambda \rangle, \quad \text{wo} \quad e_\lambda \in \Phi' \text{ gegeben.} \tag{2.5}$$

Die e_λ sind einfache gemeinsame Eigenfunktionale von (U_g):

$$U_g e_\lambda = \dot{U}_g\hat{}\,(\lambda) e_\lambda, \quad \text{genauer:} \quad \langle U_g \varphi, e_\lambda \rangle = \dot{U}_g\hat{}\,(\lambda) \langle \varphi, e_\lambda \rangle \tag{2.6}$$

identisch für $\varphi \in \Phi$. Es gelten die folgenden Formeln

$$(\dot{\varphi} \mid \dot{\psi}) = B(\varphi, \psi) = \int_\Lambda \langle \varphi, e_\lambda \rangle \overline{\langle \psi, e_\lambda \rangle}\, d\mu(\lambda) \tag{2.7}$$

oder in tensorieller Schreibweise

$$B = \int_\Lambda e_\lambda \otimes \bar{e}_\lambda\, d\mu(\lambda), \quad B \in \Phi' \otimes \Phi'. \tag{2.8}$$

$$\dot{\varphi} = \int_\Lambda \langle \varphi, e_\lambda \rangle e_\lambda\, d\mu(\lambda). \tag{2.9}$$

Bemerkung. (2.7) ist die „Parseval'sche Formel", (2.8) „Bochnerscher Satz", (2.9) die „Umkehrformel".

Aus Satz 3 folgen unmittelbar alle angekündigten Theoreme. Wir fassen sie zusammen im fo'genden

Hauptsatz. Voraussetung: *Es sei G eine l.k. kommutative separable Gruppe. T sei eine positivdefinite Distribution auf G,* d. h. *$T \in D'(G)$, $T(\varphi * \varphi^*) \geqslant 0$, wo $\varphi \in D(G)$, $\varphi^*(x) = \overline{\varphi(x^{-1})}$, $x \in G$. Bekanntlich induziert T einen p.d. hermitischen verschiebungsinvarianten Kern B auf G, $B(x, y) \overset{\text{df}}{=} T(xy^{-1})$, genauer $B(\varphi, \psi) = T(\varphi * \psi^*)$.*

Behauptung: *Dann gibt es auf der Gruppe der Charaktere $\hat{G}$ ein solches positives Mass μ_T, dass*

1° $$\langle \varphi, T\rangle = \int_{\hat{G}} \langle \varphi, \chi_\lambda\rangle \, d\mu_T(\lambda) = \int_{\hat{G}} \hat{\varphi}(\lambda)\, d\mu_T(\lambda).$$

Wenn man insbesondere $T = \varepsilon_e$ (Einheitsmass konzentriert in e) nimmt:

$T_e(\varphi * \psi^*) = (\varphi|\psi) = \int_G \varphi(x)\,\overline{\psi(x)}\, dx$, *dann hat die U_g-Fouriertransformation die „klassische" Form* $\hat{\varphi}(\lambda) = \langle \varphi, \chi_\lambda\rangle = (\varphi|\chi_\lambda) = \int_G \varphi\bar{\chi}_\lambda\, dx$; *sie ist zu einer unitären Abbildung $L^2(G) \ni \varphi \to \hat{\varphi} \in L^2(\hat{G})$ abschliessbar:*

2° $\int_G \varphi\bar{\psi}\, dx = \int_{\hat{G}} (\varphi|\chi_\lambda)\,\overline{(\psi|\chi_\lambda)}\, d\lambda$, *wobei das Mass $d\lambda$ invariant ist.*

3° *Es gilt die Umkehrformel* $\varphi(\cdot) = \int_{\hat{G}} (\varphi|\chi_\lambda)\,\chi_\lambda(\cdot)\, d\lambda$.

4° *Die stetigen Charaktere χ_λ sind gemeinsame Eigenfunktionale der Verschiebungsoperatoren:* $\chi_\lambda \in D'(G)$ $\langle U_s\varphi, \chi_\lambda\rangle = \hat{U}_s(\lambda)\langle \varphi, \chi_\lambda\rangle$. $\varphi \in D(G) = \Phi$.

5° *Die Charaktere bilden ein vollständiges System in $\Phi' = D'(G)$.*

6° *Die Charaktere trennen Punkte von G.*

Beweis. Aus dem Satz 1 und einem Satz von R. Godement [2] folgt, dass die Voraussetzungen des Satzes 3 erfüllt sind für $\Phi = B(G)$, U_g-Verschiebungsoperatoren. Da die Projektion $P : \Phi \to \Phi/N \equiv \dot{\Phi}$ stetig ist, folgt aus (1.1), dass die Abbildung $G \ni g \to P(U_g\varphi) = \dot{U}_g\dot{\varphi} \in \dot{\Phi}$ stetig für jedes $\dot{\varphi} = P\varphi \in \dot{\Phi}$ ist.

Da $\dot{U}_g\hat{}\,(\lambda)\langle \varphi, e_\lambda\rangle = \langle U_g\varphi, e_\lambda\rangle$ und $g \to U_g\varphi \in \Phi$ stetig ist, folgt aus (2.2)—(2.4), dass die Funktion $\chi_\lambda(g) \overset{\text{df}}{=} \dot{U}_g\hat{}\,(\lambda)$ stetiger Charakter von G ist. Weil

$$\langle U_g\varphi, \chi_\lambda\rangle = \int \varphi(gs)\,\overline{\chi_\lambda(g)}\, ds = \int \varphi(s)\,\overline{\chi_\lambda(g^{-1}s)}\, ds =$$

$$= \chi_\lambda(g)\langle \varphi, \chi_\lambda\rangle = \dot{U}_g\hat{}\,(\lambda) = \langle \varphi, \chi_\lambda\rangle$$

folgt aus (2.6) und aus der Einfachheit des Spektrums von $A((\dot{U}_g))$, dass $e_\lambda = a(\lambda)\,\chi_\lambda$.

Daraus und aus (2.7), (2.8) folgt, dass

$$B(\varphi, \psi) = \int_{\hat{G}} (\varphi|\chi_\lambda)\,\overline{(\psi|\chi_\lambda)}\,|a(\lambda)|\, d\mu(\lambda) = \int_{\hat{G}} \hat{\varphi}(\lambda)\,\hat{\psi}(\lambda)\, d\mu(\lambda).$$

Oder in tensorieller Schreibweise $B = \int \bar{\chi}_\lambda \otimes \chi_\lambda\, d\mu_T(\lambda)$. Da aber $(\chi_\lambda \otimes \bar{\chi}_\lambda)(x, y) = \chi_\lambda(x)\,\overline{\chi_\lambda(y)} = \chi_\lambda(xy^{-1})$, $B(x, y) = T(xy^{-1}) = \int \chi_\lambda(xy^{-1})\, d\mu_T(\lambda)$.

Auf diese Weise sind 1°—5° bewiesen. Man zeigt ohne grössere Mühe, dass das Plancherel-Mass $d\mu_{\varepsilon_0}(\lambda) = d\lambda$ invariant ist.

Da man das Spektrum Λ von $A((U_g))$ mit $\hat{G} \cup \{0\}$ identifizieren kann, gibt es in g_1, g_2, $(g_1 \neq g_2)$ ein solches $\lambda \in \Lambda$, dass $\chi_\lambda(g_1) = U_\lambda\hat{}\,(g_1) \neq U_\lambda\hat{}\,(g_2) = \chi_\lambda(g_2)$, so dass auch 6° bewiesen ist.

KOROLLAR. *(Dualitätssatz von Pontriagin) folgt auf übliche Weise.*

Diese Arbeit ist dem Gedächtnis von Hidehiko Yamabe gewidmet.

Bemerkung bei Korrektur; In einer eben erschienenen wichtigen Abhandlung [7] hat G. I. Kac die Sätze 1 und 2 auf andere Weise bewiesen.

MATHEMATISCHES INSTITUT, POLNISCHE AKADEMIE DER WISSENSCHAFTEN
(INSTYTUT MATEMATYCZNY, PAN)

SCHRIFTTUM

[1] I. Gelfand, N. Wilenkin, *Verallgemeinerte Funktionen*, Bd 4, Moskau, 1961 [russisch].

[2] R. Godement, *Théorie des caractères* II. *Définition et propriétés générales des caractères*, Ann. of Math., **59** (1954), 63.

[3] G. I. Kac, *Verallgemeinerte Funktionen auf l.k. Gruppen und eine Zerlegung der regulären Darstellung*, Dokł. Akad. Nauk, **125** (1959), 27—30 [russisch].

[4] K. Maurin, *Allgemeine Eigenfunktionsentwicklungen. Spektraldarstellung abstrakter Kerne. Eine Verallgemeinerung der Distributionen auf Lie'schen Gruppen*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **7** (1959), 471—479.

[5] — , *Elementare Bemerkungen über kommutative C*-Algebren*, Studia Math., **16** (1957), 74.

[6] A. Weil, *Intégration dans les groups topologiques et ses applications*, Paris, 1940.

[7] G. I. Kac, *Verallgemeinerte Funktionen auf einer l. k. Gruppe und Zerlegungen unitärer Darstellungen* [russisch] Trudy Mosk. Mat. Obšč., **10** (1961), 1—40.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 12, 1961

MATHEMATICS

Stability of the Index of Elliptic Boundary Value Problems

by

K. MAURIN

Presented by S. MAZUR on September 23, 1961

In the present note it is proved that the index of the elliptic boundary problems is not changed by the perturbations of lower order.

Let H be a complex Banach space, H^* — its strong dual and A — a linear densely defined operator in H with the domain $D(A)$ and the range $R(A)$.

$\alpha(A) = \dim\{u \in H : Au = 0\}$, $\beta(A)$ is the codimension of $R(A)$.

When $\alpha(A), \beta(A) < \infty$, then the number $\varkappa(A) = \beta(A) - \alpha(A)$ is called the index of the operator A.

The differential operator $a(x, D)$ of order $2m$ in Ω_n is the expresion $a(x, D) = \sum_{|\alpha| \leqslant 2m} a_\alpha(x) D^\alpha$, where $D_j = i^{-1}\,\partial/\partial x_j$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $x \in \Omega_n \subset E^n$.

The characteristic form of $a(x, D)$ is $a(x, \xi) = \sum_{|\alpha| = 2m} a_\alpha(x)\,\xi^\alpha$,

$$\xi = (\xi_1, \ldots, \xi_n), \qquad \xi_n \in \varphi^1, \qquad \xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}.$$

If the $B_1, \ldots, B_m$ are differential operators of order $< 2m$ defined on a neighbourhood of the regular boundary $\partial\Omega_n$ of Ω_n, each B_j may be written as $\sum b_{j,\beta}(x)\, D^\beta$, $|\beta| \leqslant r$; and its characteristic form as $b_j(x, \xi) = \sum b_{j,\beta}(x)\,\xi^\beta$, $|\beta| = r_j$, where r_j is the order of B_j.

DEFINITION. The system $(A, B_1, \ldots, B_m)$ is called elliptic boundary problem, if for every $x \in \partial\bar{\Omega}_n$ there exists a constant $c(x) > 0$, such that $|\det(p_{jk}(x, t))| \geqslant c(x) > 0$ for every vector t tangent to $\partial\Omega_n$ with $\|t\| = 1$.

1° The polynomial $a(x, t+\lambda n)$ in the complex variable λ has exactly m roots with positive imaginary part and

2° if $p_{j,k}(x, t) = \int_{C_t} \lambda^{j-1}\, b_k(x, t+\lambda n)\,(a(x, t+\lambda n))^{-1}\, d\lambda$,

where C_t is a curve in the upper half-plane surrounding the zeros of $a(x, t+\lambda n)$; $j, k = 1, 2, \ldots, m$; then there exists $c(x) > 0$, such that $|\det(p_{j,k}(x, t+\lambda n))| \geqslant c(t) > 0$ for all vectors t tangent to $\partial\Omega_n$ at x with $\|t\| = 1$, where n denotes the normal vector to $\partial\Omega_n$ at x with $\|n\| = 1$.

F. E. Browder proved in [1] that the elliptic operator considered as an operator in the Banach space $H = L^p(\Omega_n)$ with the domain $D(A) = \{u : D^\alpha u \in L_p(\Omega_n), |\alpha| \leqslant 2m, B_j u = 0 \text{ for } 1 \leqslant j \leqslant m\}$, $p > 1$ — where derivatives are taken in the distribution sense — is a closed operator with finite defect indices.

We can now prove the following

THEOREM. *Let $(A, B_1, \ldots, B_m)$ be an elliptic boundary problem of order $2m$ and let T be a differential operator of order $\leqslant 2m-1$ with bounded and measurable coefficients on Ω_n. Then the index of the operator $A+T$ considered as an operator in $L^p(\Omega_n)$ is the same as the index of A : $\varkappa(A+T) = \varkappa(A)$.*

Proof. The operator A being closed, its domain $D(A)$ is a Banach space in the graph norm $|u| = \|u\| + \|Au\|$.

The operator T considered as a mapping of this Banach space into $L^p(G)$ is compact or in a recent terminology is a-compact. Indeed,

$$(1) \qquad \|Tu\|_0 \leqslant M\|u\|_{2m-1}, \quad \text{where} \quad M = \sup_\alpha |t_\alpha(\cdot)| \quad \text{and} \quad \|u\|_k^p = \int_{\Omega_n} \sum_{|\alpha| \leqslant k} |D^\alpha u|^p \, dx.$$

But for the elliptic boundary problem the fundamental inequality holds

$$(2) \qquad \|u\|_{2m} \leqslant c(\|Au\|_0 + \|u\|_0) = c|u|.$$

Therefore the norms $\|\ \|_{2m}$ and $|\cdot|$ are equivalent and a set $|\ |$-bounded is $\|\ \|_{2m}$-bounded.

From the Rellich—Sobolev lemma it follows, however, that every $\|\ \|_{2m}$-bounded set is $\|\ \|_{2m-1}$ precompact. Thus T is A-compact.

But, as is well known (comp. [2]), $\varkappa(A+T) = \varkappa(A)$ for every A-compact operator T. Thus the Theorem is proved.

DEFINITION. The complex number λ_0 is called the Φ-point of the closed operator A if $(A - \lambda I)$ has finite indices of defect.

From our Theorem follows the

COROLLARY. *For the elliptic boundary problem every complex number is a Φ-point of the operator A.*

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY, PAN)

REFERENCES

[1] F. E. Browder, *Estimates and existence theorems for elliptic boundary value problems*, Proc. Nat. Nat. Acad. Sci., **45** (1959), 365—372.

[2] B. z. Nagy, *On the stability of the index of unbounded linear transformations*, Acta Math. Sci. Hung., **3** (1952), 49—52.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 12, 1961

MATHEMATICS

Note on a Question of Malcev

by

S. MORAN

Presented by K. KURATOWSKI on September 27, 1961

Shmelkin [10] has defined a regular multiplication*) in the class of all groups which has the following properties:

(i) Commutative.

(ii) Homomorphisms of factors of the regular product extend uniquely to give a homomorphism of the regular product.

(iii) A regular product cannot be decomposed into a free product which is not the infinite dihedral group.

(iv) If in a regular product a subgroup is taken from each of the regular factors, then these subgroups generate a group isomorphic to the same regular product of these subgroups.

This multiplication of Shmelkin is defined by**)

$$\prod_{\alpha\in M}^{\circledcirc} G = F/V([G_\alpha]^F),$$

where G_α, $\alpha\in M$, is an arbitrary set of groups, F denotes their free product, $[G_\alpha]^F$ is the cartesian subgroup and $V(X)$ denotes a verbal subgroup. Direct and free multiplications are particular cases of this multiplication. In fact, they are the only cases in which $\circledcirc$ — multiplication is known to be an associative operation.

Struik [11] has given examples of non-associative $\circledcirc$ — multiplications.

Malcev has raised the question: Do there exist associative regular multiplications, other than direct and free multiplications, which satisfy the above properties (i) and (iv)? The answer to this question is not known to the author, but the available information seems to suggest that the answer is no. However, we give an example of an associative regular multiplication which satisfies properties (i) and (ii) and

(iii)′ If a regular product is not equal to the free product of the same factors, then it cannot be decomposed into a free product;

*) Cf. Golovin [1] for the concept of regular multiplication.

**) Gruenberg was the first to consider $\circledcirc$-products, cf. [2], Theorem 6.2.

(iv)′ If in a regular product a *serial* subgroup is taken from each of the regular factors, then these subgroups generate a group isomorphic to the same regular product of these subgroups.

A subgroup H of a group G is said to be serial if it is possible to ascend from H to G by means of a well ordered ascending normal system (cf. Kuroš [4]). In particular, every normal and subnormal subgroup is serial. An element of a group is said to be serial, if the subgroup generated by this element is a serial subgroup. Gruenberg [3] has shown that the set of all serial elements of an arbitrary group G forms a characteristic subgroup $\sigma(G)$ of G. We now proceed to show that the N^*-multiplication, as defined in [6], with $N(X) = \sigma(X)$ has the above given properties.

Firstly we note that $\sigma(X)$ is a strongly characteristic subgroup function, that is,

$$\sigma(X)\,\Phi \leqslant \sigma(X\Phi)$$

for every homomorphism Φ of X. If F is a free product which is not the infinite dihedral group, then

$$\sigma(F) = 1.$$

This is the direct consequence of the following two facts.

By Gruenberg ([3], Theorem 2) $\sigma(X)$ is contained in the Hirsch—Plotkin radical*). By a result of [7], the Hirsch—Plotkin radical of F is the trivial group. Also

$$\sigma(G_\alpha) < \sigma\Big(\prod_{\alpha\in M}{}^{\sigma^*} G_\alpha\Big)$$

for all α of M. Hence, by [6] (Theorem 16.14), the σ^*-multiplication is an associative regular multiplication. Properties (i) and (iii)′ hold for all N^*-multiplications. By [8] (Theorem 1.11), if $N(X)$ is a strongly characteristic subgroup function, then the corresponding N^*-multiplication satisfies property (ii). It remains to consider property (iv)′. By [6] (Theorem 16.10)

$$[G_\alpha]^{N^*} \cong [G_\alpha/N(G_\alpha)]^E = [(G_\alpha/N(G_\alpha))^E],$$

where E is the free product of the groups $G_\alpha/N(G_\alpha)$, $\alpha \in M$. Now it is easy to see that property (iv)′ holds for σ^*-multiplication because of the following simple fact. If Y is a serial subgroup of X, then

$$\sigma(X) \cap Y = \sigma(Y).$$

Verbal multiplication does not in general satisfy property (iv). This is shown by the fact that (cf. [5], Corollary 4.2.2)

$$\{V(A),\ V(B)\} = V(A)\times V(B).$$

Suppose that property (iv)′ holds for some N-multiplication and we are able to establish this fact in the above manner. Then the fact that Y is a normal subgroup of X must imply that

$$N(X) \cap Y = N(Y).$$

*) The maximal normal locally nilpotent subgroup.

Hence, in particular,

$$N(F) \cap [G_\alpha]^F = N([G_\alpha]^F).$$

Let us confine our attention to the case when $N(X)$ is a strongly characteristic subgroup function. By B. H. Neumann ([9], Lemma 2.2), the value of a strongly characteristic subroup function on a free group of infinite tank is a verbal subgroup of this free group. Thus, at least in the case of infinite groups, our N-multiplication is nothing other than Shmelkin's $\bigcirc$-multiplication.

DEPARTMENT OF MATHEMATICS, UNIVERSITY, GLASGOW (SCOTLAND)

REFERENCES

[1] O. N. Golovin, *Nilpotent products of groups*, Mat. Sbornik, **27** (69) (1950), 427—454.

[2] K. W. Gruenberg, *Residual properties of infinite soluble groups*, Proc. London Math. Soc., **7** (1957), 29—62.

[3] — , *The Engel elements of a soluble group*, Illinois J. of Math., **3** (1959), 151—168.

[4] A. G. Kuroš, *Theory of groups*. Second Edition, 1953. English translation, 1956, New York.

[5] S. Moran, *Associative operations on groups I*, Proc. London Math. Soc., **6** (1956), 581—596.

[6] — , *Associative operations on groups III*, ibid., **9** (1959), 287—317.

[7] — , *Associative regular operati ns on groups corresponding to a property P*, Proc. American Math. Soc., **10** (1959), 796—799.

[8] — , *Properties of N-multiplications and N*-multiplications*, J. London Math. Soc. **36** (1961), 193—210.

[9] B. H. Neumann, *Ascending verbal and Frattini series*, Math. Zeit., **69** (1958), 164—172.

[10] A. L. Shmelkin, *On the theory of regular products of groups*, Mat. Sbornik, **51** (93), (1960), 277—292.

[11] R. R. Struik, *On associative products of groups*, Trans. American Math. Soc., **81** (1956), 425—452.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 12, 1961

MATHEMATICS

Remark on Measurable Set-valued Functions

by

A. PLIŚ

Presented by T. WAŻEWSKI on October 7, 1961

THEOREM 1. *Let $F(x)$ be a function defined on a cube A with the range in the family of non-empty closed subsets of a given compact metric space S. Let us assume that the set*

(1) $(x : F(x) \subset Z)$ *is measurable for every closed set* $Z \subset S$.

Then for any positive number r there exists such a closed subset B of A that $F(x)$ is continuous on B, where in the space of sets $Z \subset S$ Hausdorff's distance is used, and the measure of A—B is less than r.

Theorem 1 is an analogue of Lusin's theorem. The assumptions of Theorem 1 are formulated by means of inclusion (property (1)) and therefore Theorem 1 is convenient for applications to semicontinuous set-valued functions (Theorem 2).

Proof of Theorem 1. From (1) it follows that the set

(2) $(x : F(x) \cap W \neq \emptyset)$ is measurable for every open set $W \subset S$,

where $\emptyset$ denotes the empty set, because $(x : F(x) \cap W \neq \emptyset) = A - (x : F(x) \subset S - W)$ and the set $S - W$ is closed.

From compactness of S follows the existence of open sets U_{jk}, $j = 1, 2, \ldots$, $k = 1, \ldots, p_j$, satisfying the following conditions

$$S = \sum_{k=1}^{p_j} U_{jk}, \quad j = 1, 2, \ldots, \tag{3}$$

$$\text{diameter of } U_{jk} < 1/j, \quad j = 1, 2, \ldots, \quad k = 1, \ldots, p_j. \tag{4}$$

Let $K_{j1}, \ldots, K_{jq_j}$, $q_j = 2^{p_j} - 1$, be the array of all sets of (one up to p_j) indices $1, \ldots, p_j$. The sets

$$A_{jm} = (x : F(x) \subset \sum_{k \in K_{jm}} \overline{U}_{jk} \quad \text{and} \quad F(x) \cap U_{jk} \neq \emptyset \quad \text{for} \quad k \in K_{jm}), \tag{5}$$

where $\overline{U}_{jk}$ denotes the closure of U_{jk}, are measurable in virtue of (1), (2). Moreover, we have

$$A = \sum_{k=1}^{q_j} A_{jk}, \quad j = 1, 2, \ldots . \tag{6}$$

Indeed, let y be an arbitrary point of A. $F(y)$ is non-empty and $F(y) \subset S$. Therefore from (3) follows the existence of an integer n satisfying the condition $F(y) \cap U_{jn} \neq \emptyset$.

Let K_{jm} be the set of all indices n satisfying this condition. From (3) we have $F(y) \subset \sum_{k \in K_{jm}} U_{jk}$ and hence $y \in A_{jm}$, in virtue of (5).

From (4), (5) it is easy to obtain the inequality

$$h(F(x), F(y)) \leqslant 1/j \quad \text{for} \quad x, y \in A_{jk}, \quad j = 1, \ldots, \quad k = 1, \ldots, q_j, \tag{7}$$

where $h(U, V)$ denotes Hausdorff's distance of sets U, V.

Let r_{ij}, $i, j = 1, 2, \ldots$, be such positive numbers that

$$\sum_{i,j=1}^{\infty} r_{ij} < r. \tag{8}$$

Denote by B_{jk}, C_{jk} such closed subsets of A that

$$B_{jk} \subset A_{jk}, \tag{9}$$

$$C_{jk} \subset A - A_{jk}, \tag{10}$$

$$\mu(D_{jk}) < r_{jk}, \tag{11}$$

where

$$D_{jk} = A - (B_{jk} + C_{jk}), \tag{12}$$

$j = 1, \ldots, k = 1, \ldots, q_j$, and $\mu(D)$ denotes the measure of the set D. In virtue of (9), (10) $B_{jk} \cap C_{jk} = \emptyset$ and hence there exists such a sequence e_j that

$$d(B_{jk}, C_{jk}) > e_j > 0, \quad j = 1, 2, \ldots, \quad k = 1, \ldots, q_j, \tag{13}$$

where $d(B, C) = \inf |x - y|$, $x \in B$, $y \in C$. Denote

$$B = A - \sum_{j=1}^{\infty} \sum_{k=1}^{q_j} D_{jk}. \tag{14}$$

From (12) it follows that B is closed. In virtue of (12), (14) we have $B - B_{jk} \subset C_{jk}$ and therefore from (13) we obtain

$$d(B_{jk}, B - B_{jk}) > e_j > 0, \quad j = 1, \ldots, \quad k = 1, \ldots, q_j. \tag{15}$$

From (12), (10) it follows that $A_{jk} - B_{jk} \subset D_{jk}$ and consequently in virtue of (14), (6) we obtain

$$B \subset \sum_{k=1}^{q_j} B_{jk}, \quad j = 1, 2, \ldots . \tag{16}$$

The inequality $\mu(A - B) < r$ results from (14), (11), (8).

To prove the continuity of $F(x)$ on B we shall now show that if $x, y \in B$, $|x - y| < e_j$, then $h(F(x), F(y)) \leqslant 1/j$, $j = 1, 2, \ldots$. In virtue of (16) there exists

for any j an integer m such that $x \in B_{jm}$. Hence, in virtue of (15) and of the inequality $|x - y| < e_j$, the relation $y \in B - B_{jm}$ is not satisfied and therefore $y \in B_{jm}$. From (9), (7) it follows that $h(F(x), F(y)) \leqslant 1/j$. Thus the proof of Theorem 1 is complete.

If a function $F(x)$ is lower semicontinuous (i.e. if for every $x, u \in F(x)$ implies the existence of such a function $w(y)$ defined in a neighbourhood of x, restricted to the domain of $F(x)$, that $w(y) \subset F(y)$ and $w(y) \to u$ as $y \to x$) the set (1) is closed provided Z is closed and hence we obtain the following corollary:

THEOREM 2. *For every lower semi-continuous set-valued function $F(x)$ defined on a cube A with the range in the family of non-empty closed subsets of a given compact metric space S and any positive number r there exists such a closed subset of A that $F(x)$ is continuous on B and the measure of $A - B$ is less than r.*

Theorem 2 is an answer to a problem posed by T. Ważewski.

INSTITUTE OF MATHEMATICS, CRACOW BRANCH, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY, ODDZIAŁ KRAKÓW, PAN)

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 12, 1961

MATHÉMATIQUE

Sur une condition d'existence des fonctions implicites mesurables

par
T. WAŻEWSKI

Présenté le 10 octobre 1961

Le Théorème 2 donné plus bas donne une certaine condition suffisante pourqu'il existe au moins une fonction implicite mesurable $s(t)$ satisfaisant aux relations (4.1). Ce Théorème se rattache de prés à un problème relatif à l'équivalence des systèmes de commande et des équations au contingent (cf. [2], p. 153). Le Théorème 2 résulte d'un théorème plus général (Théorème 1) dont nous présentons la démonstration. On pourrait le démontrer aussi par une autre voie en utilisant l'idée appliquée dans notre travail antérieur ([1], pp. 261—263).

I. NOTATIONS. R_q designe l'espace euclidien réel à q dimensions, $r(a, b)$ — la distance euclidienne de $a \in R_q$, $b \in R_q$. Le voisinage fermé (de rayon $k > 0$) de l'ensemble $A \subset R_q$ est designé par $V(A, k)$. La relation $A \in \mathrm{comp}(R_q)$ veut dire que l'ensemble A est non vide, compact et $A \subset R_q$.

DÉFINITION 1. Soit $B \subset R_q$, $N(w) \in \mathrm{comp}(R_p)$ lorsque $w \in B$. Si à tout $w_0 \in B$ et $k > 0$ correspond un $m > 0$ tel que $N(w) \subset V(N(w_0), k)$ lorsque $w \in BV(w_0, m)$, on dit que $N(w)$ est sémicontinu par inclusion dans B et on peut écrire $N(w) \in \mathrm{sém.\ incl.}(B)$.

DÉFINITION 2. On dit que $N(w)$ est borné dans B lorsqu'il existe un K indépendant de w tel que $K \in \mathrm{comp}(R_p)$ et $N(w) \subset K$ pour $w \in B$.

On établit facilement les propositions suivantes:

PROPOSITION 1. *Si* $B \in \mathrm{comp}(R_q)$, $N(w) \in \mathrm{sém.\ incl.}(B)$, $N(w) \in \mathrm{comp}(R_p)$ *pour* $w \in B$, *alors* $N(w)$ *est borné dans* B.

PROPOSITION 2. *Si* $B \subset R_q$, $N(w) \in \mathrm{comp}(R_q)$ *pour* $w \in B$ *et* $N(w)$ *est borné dans* B, *alors la condition nécessaire et suffisante pour que* $N(w) \in \mathrm{sém.\ incl.}(B)$ *consiste en ce que pour tout* $w_0 \in B$ *les relations* $w_n \in B$ $w_n \to w_0$ $v_n \in N(w_n)$, $v_n \to v_0$ *entraînent la relation* $v_0 \in N(w_0)$.

II. DÉFINITION 3. Soit $M \in \mathrm{comp}(R_p)$. Nous désignons par $m_j(M)$ le minimum de l'ensemble des j-èmes coordonnées des $u \in M$.

Remarque 1. Si $M \in \mathrm{comp}(R_p)$, alors $m_j(M)$ existe et il existe un point $u \in M$ dont la j-ème coordonnée est égale à $m_j(M)$.

III. Hypothèse $K(N, B)$. $B \in \mathrm{comp}\,(R_q)$, $N(w) \in \mathrm{comp}\,(R_p)$ *pour* $w \in B$, $N(w) \in$ sém. incl. (B).

Remarque 2. Sous l'hypothèse $K(N, B)$ les fonctions $m_j(N(w))$ $(j = 1, ..., p)$ sont sémicontinues inférieurement dans B, donc mesurables au sens de Lebesgue dans B.

Définition 4. Admettons l'hypothèse $K(N, B)$. Nous allons définir le point

$$L(N(w)) = (l_1(N(w)), ..., l_p(N(w))). \tag{3.1}$$

Nous posons (cf. Déf. 3)

$$l_1(N(w)) = m_1(N(w)). \tag{3.2}$$

Les fonctions

$$l_1(N(w)), ..., l_{k-1}(N(w)), \quad (2 \leqslant k \leqslant p) \tag{3.3}$$

étant supposées définies nous envisageons dans R_p l'hyperplan $P_{k-1}(w)$ à $p - k + 1$ dimensions

$$u_1 = l_1(N(w)), ..., u_{k-1} = l_{k-1}(N(w)) \tag{3.4}$$

et nous posons

$$N_k(w) = N(w) \cdot P_{k-1}(w), \tag{3.5}$$

$$l_k(N(w)) = m_k(N_k(w)). \tag{3.6}$$

Le point (3.1) ainsi défini est dit minimum lexicographique de $N(w)$. Si nous remplaçons $N(w)$ par N fixe $(N \in \mathrm{comp}\,(R_p))$, alors le plan (3.4) sera fixe et la formule (3.1) fournira le minimum lexicographique de N.

Théorème 1. *Sous l'hypothèse* $K(N, B)$

1°) *le point* $L(N(w))$ *(minimum lexicographique de* $N(w)$*) existe et*

$$L(N(w)) \in N(w) \quad \textit{pour} \quad w \in B \tag{3.7}$$

2°) *la fonction* $L(N(w))$ *est mesurable dans* B*, c'est-à-dire les fonctions* $l_i(N(w))$ (cf. (3.1)) *sont mesurables au sens de Lebesgue dans* B.

Démonstration. L'assertion 1° est évidente. La fonction (3.2) est mesurable (cf. Remarque 2). Admettons que les fonctions (3.3) soient mesurables dans B. Soit $k > 0$. Il existe donc, en vertu d'un théorème de Egoroff, un ensemble $B(k)$, tel que les fonctions (3.3) envisagées dans $B(k)$ sont continues et $B(k) \subset B$, $B(k) \in \mathrm{comp}\,(R_p)$, mesure $(B - B(k)) \leqslant k$. En vertu de la Proposition 2, on conclue facilement que (cf. 3.5) $N_k(w) \in$ sém. incl. $(B(k))$ d'où il résulte (cf. Remarque 2 avec $B(k)$, $N_k(w)$, au lieu de B, $N(w)$) que la fonction (3.6) est mesurable dans $B(k)$. Le passage à la limite $k \to 0$ conduit à la conclusion que la fonction (3.6) est mesurable dans B.

IV. Théorème 2. Prémisses. $C \in \mathrm{comp}\,(R_p)$, $Q \in \mathrm{comp}\,(R_n)$, $J \in \mathrm{comp}\,(R_1)$. *Pour les variables* x, u, t *on a* $x \in R_n$, $u \in R_p$, $t \in R_1$.

$C(t, x) \in \mathrm{comp}\,(R_p)$, $C(t, x) \subset C$ *pour* $(t, x) \in J \times Q$. $C(t, x) \in$ sem. incl. $(J \times Q)$. *La fonction* $f(t, x, u) = (f_1(t, x, u), ..., f_n(t, x, u))$ *est continue dans* $J \times Q \times C$ *et.* $Z \subset J$ *désigne un ensemble de mesure nulle.* $v = x'(t) = dx(t)/dt$ *est mesurable*

dans $J - Z$ *et* $u = u(t)$ *est definie dans* $J - Z$ *sans y être forcément mesurable. On a*

$$x'(t) = f(t, x(t), u(t)), \quad u(t) \in C(t, x(t)) \quad \text{pour} \quad t \in J - Z.$$

THÈSE. *Il existe une fonction* $u = s(t)$ *définie presque partout dans* J *et mesurable dans* J, *telle que presque partout dans* J *on a*

$$x'(t) = f(t, x(t), s(t)), \quad s(t) \in C(t, x(t)). \tag{4.1}$$

Démonstration. Posons pour $t \in J - Z$

$$N(t) = \{u : x'(t) = f(t, x(t), u), \quad u \in C(t, x(t))\}.$$

On a $N(t) \in \operatorname{comp}(R_p)$ pour $t \in J - Z$. Posons (cf. (3.1)) $s(t) = L(N(t))$. La relation (4.1) a lieu. Soit $k > 0$. Il existe un ensemble compact $B(k)$ $J - Z$ tel que la mesure $(J - Z - B(k)) \leqslant k$ et tel que $x'(t)$ et $x(t)$ sont continues lorsque l'on les envisage exclusivement dans $B(k)$ On a (cf. Propos. 2) $N(t) \in \text{sem. incl.}\, B(k)$ donc (cf. Théor. 1) $s(t) = L(N(t))$ est mesurable dans $B(k)$.

Le passage à la limite $k \to 0$ achève la démonstration.

INSTITUT MATHÉMATIQUE, ACADÉMIE POLONAISE DES SCIENCES
(INSTYTUT MATEMATYCZNY, PAN)

OUVRAGES CITÉS

[1] T. Ważewski, *Sur les jacobiens asymptotiques et le changement de variables dans les intégrales multiples*, Bull. de l'Acad. Polon. des. Sci. et des Lettres, Cl. Sci. Math. et Natur. Sér. A. Sci. Math., Kraków (1930), 249—299.

[2] — , *Systèmes de commande et équations au contingent*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **9** (1961), 151—155.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 12, 1961

MATHÉMATIQUE

Sur une condition équivalente à l'équation au contingent

par

T. WAŻEWSKI

Présenté le 10 octobre 1961

Nous prouvons (cf. Théorème 1) que l'équation au contingent (3.3) au sens de M. Marchaud et S. K. Zaremba est équivalente à la condition (3.5) dans laquelle la dérivée contingentielle n'intervient pas, étant remplacée par la dérivée ordinaire $x'(t)$. Relativement à la fonction inconnue $x(t)$ intervenant dans (3.3), on admet qu'elle est continue, tandis que relativement à $x'(t)$ intervenant dans (3.5) on suppose qu'elle est absolument continue. Il est à souligner que l'on exige que (3.3) ait lieu partout dans un intervalle, tandis qu'il suffit d'exiger que (3.5) ait lieu presque partout.

Le Théorème 1 joue un rôle essentiel dans la démonstration de l'équivalence du système de commande S envisagé dans notre note [1] à l'equation au contingent S^* associée à S.

Remarquons enfin que dans la formule (S^*) ([1], p. 153) une erreur évidente s'est introduite. La formule (S^*) doit avoir la forme (3.3) de la présente note.

NOTATIONS ET DÉFINITIONS.

I. R_n désigne l'espace euclidien réel à n dimensions, $R_p \times R_q$ le produit cartésien, $|x|$ la longueur euclidienne du vecteur $x \in R_n$. La relation $A \in \mathrm{comp}(R_n)$ veut dire que $A \subset R_n$ est compact et non vide. Par $V(A, k)$ on désigne le voisinage fermé de l'ensemble A de rayon $k > 0$. t désigne le temps.

II. J désigne un intervalle ($J \subset R_1$). La fonction

$$x = x(t) = (x_1(t), \ldots, x_n(t)), \qquad (x(t) \in R_n) \tag{1.1}$$

est dite absolument continue dans J lorsque les fonctions composantes $x_i(t)$ sont absolument continues dans tout sousintervalle compact de J.

III. Soit $t_0 \in J$ et posons $p(t) = (x(t) - x(t_0))/(t - t_0)$. L'ensemble $M \subset R_n$ sera dit dérivée contingentielle de $x(t)$ à l'instant t_0 et sera désigné par $D^* x(t_0)$ lorsque, pour $t \to t_0$, on a $\limsup |p(t)| < +\infty$ et lorsque M constitue la classe de tous les $c \in R_n$ pour lesquels il existe une suite $\{t_j\}$, telle que $t_j \in J$, $t_j \neq t_0$, $t_j \to t_0$, $p(t_j) \to c$.

IV. Soit $z > 0$. Par réseau $G(z)$ nous entendons la subdivision de R_n en mailles disjointes revêtant la forme de hypercubes fermés à gauche et ouverts à droite, définis par les inégalités $q_i z \leqslant x_i < (q_i+1) z$, $(i = 1, \ldots, n)$ où q_i est un nombre entier quelconque.

V. Soit $B \subset R_q$, $N(w) \in \mathrm{comp}(R_p)$ pour $w \in B$. Il peut arriver que pour chaque $w_0 \in B$ et $k > 0$ il existe un $u > 0$, tel que $N(w) \subset V(N(w_0), k)$ lorsque $w \in BV(w_0, u)$. On dit alors que $N(w)$ est sémicontinu par inclusion dans B et on écrit $N(w) \in \text{sém. incl.}(B)$.

Lemme 1. *Si $S \in \mathrm{comp}(R_n)$, l'ensemble S est convexe, $a < b$, la fonction $x(t)$* (cf. (1.1)) *est absolument continue dans $[a, b]$ et $x'(t) \in S$ pour presque tous les $t \in [a, b]$, alors $((x(b) - x(a))/(b - a)) \in S$.*

Démonstration. Posons $W = (\int_a^b x'(t)\, dt)/(b-a) = (x(b) - x(a))/(b-a)$. Soit $k > 0$. Choisissons $z > 0$ de façon à ce que le diamètre des mailles du réseau $G(z)$ (cf. IV) ne surpasse pas k. Soient $M_1, \ldots, M_q$ les mailles de $G(z)$ empiétant sur S. Posons $P_i = S \cdot M_i$. On a le diamètre $(P_i) \leqslant k$, $\sum P_i = S$. Soit $T_i = \{t : t \in [a, b], x'(t) \in P_i\}$. Les T_i sont mesurables et disjoints. Soit m_i la mesure de T_i. On a $\sum m_i = b - a$. Posons $h_i = m_i/(b-a)$. Choisissons un $r_i \in P_i \subset S$. Comme $h_i \geqslant 0$, $\sum h_i = 1$ et S est convexe et compact on a

$$\sum h_i r_i \in S. \tag{2.1}$$

Étant donné que le diamètre $(P_i) \leqslant k$, on a $|\int_a^b x'(t)\, dt - \sum m_i r_i| \leqslant k(b-a)$ d'où il vient que $|W - \sum h_i r_i| \leqslant k$. Il s'ensuit en vertu de (2.1) (cf. I) que $W \in V(S, k)$. En effectuant le passage à la limite $k \to 0$, on en conclut que $W \in S$.

Théorème 1. Prémisses. *(t, x) désigne le point variant dans un ensemble ouvert $W \subset R_1 \times R_n$. L'ensemble $E(t, x)$ est convexe et $E(t, x) \in \mathrm{comp}(R_n)$ pour $(t, x) \in W$. On a* (cf. V)

$$E(t, x) \in \text{sém. incl.}(W). \tag{3.1}$$

J désigne un intervalle.

Thèse. *Le système des conditions* (cf. les notations III et II)

$$x(t) \text{ est continue dans } J, \tag{3.2}$$

$$D^* x(t) \subset E(t, x(t)) \text{ partout dans } J \tag{3.3}$$

est équivalent au système des conditions suivantes:

$$x(t) \text{ est absolument continue dans } J, \tag{3.4}$$

$$x'(t) \in E(t, x(t)) \text{ presque partout dans } J. \tag{3.5}$$

Démonstration. Admettons que (3.2) et (3.3) aient lieu et soit L un sous-intervalle compact de J. De (3.1) et (3.2) il résulte que $E(t, x(t)) \in \text{sém. incl.}(L)$. Il

s'ensuit qu'il existe un $C \in \mathrm{comp}(R_n)$ tel que $D^* x(t) \subset E(t, x(t)) \subset C$ pour $t \in L$. On en conclut que $x(t)$ satisfait à la condition de Lipschitz dans L et, par suite, $x(t)$ est absolument continue dans L d'où résulte (3.4) (cf. II). La dérivée $x'(t)$ existe donc presque partout dans J. En observant que

$$x'(t) \in D^* x(t) \subset E(t, x(t)) \text{ presque partout dans } J$$

on obtient (3.5).

Admettons maintenant (3.4) et (3.5). La condition (3.2) est donc satisfaite Soit $t_0 \in J$, $k > 0$ et posons (cf. I) $U(k) = V(E(t_0, x(t_0)), k)$. On a $U(k) \in \mathrm{comp}(R_n)$ et $U(k)$ est convexe. En vertu de (3.1) on a $E(t, x)(t)) \in \text{sém. incl.}(J)$. Il existe donc (cf. V) un voisinage H de t_0 tel que $E(t, x(t)) \subset U(k)$ pour $t \in H$. On a donc (cf. (3.5)) $x'(t) \in U(k)$ presque partout dans H, ce qui—rapproché de (3.4) — conduit, en vertu du Lemme 1, à la conclusion que $(x(t) - x(t_0))/(t - t_0) \in U(k)$ lorsque $t \neq t_0$, $t \in H$.

Il s'ensuit (cf. III) que $D^* x(t_0) \subset U(k)$ d'où, par le passage à la limite $k \to 0$, on obtient $D^* x(t_0) \subset E(t_0, x(t_0))$. La relation (3.3) a donc lieu.

INSTITUT MATHÉMATIQUE, ACADÉMIE POLONAISE DES SCIENCES
(INSTYTUT MATHEMATYCZNY, PAN)

OUVRAGES CITÉS

[1] T. Ważewski, *Système de commande et équations au contingent*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **9** (1961), 151—155.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 12, 1961

MATHÉMATIQUE

Sur la sémicontinuité inférieure du „tendeur" d'un ensemble compact variant d'une façon continue

par
T. WAŻEWSKI

Présenté le 10 octobre 1961

Soit A un ensemble compact et E son enveloppe convexe. Nous donnons dans la suite une définition (cf. Déf. 2) du tendeur de A, que nons désignons par tend (A); il constitue un sousensemble de A. Nous allons prouver que tend (A) = tend (E). Les résultats de la présente note (cf. en particulier Théor. 4) constituent donc une contribution à la théorie des ensembles convexes.

NOTATIONS ET REMARQUES PRÉLIMINAIRES

I. R_n désigne l'espace cartésien à n dimensions, l'expression $A \in \text{comp}(R_n)$ veut dire que $A \subset R_n$ et que A est compact et non vide. ferm (A) désigne la fermeture de A. $E \in \text{conv}$ exprime que E est convexe, compact et non vide.

II. Soit $p \in E$, $E \in \text{conv}$. On dit que p est un point extrême de E lorsqu'il n'est situé à l'intérieur d'aucun segment appartenant à E. La classe de tous les p satisfaisant à cette condition sera désignée par extr (E). Nous posons

$$\text{tendeur de } E = \text{tend}(E) = \text{ferm}(\text{extr}(E)). \tag{1.1}$$

On a

$$\text{extr}(E) \subset \text{tend}(E) \subset E. \tag{1.2}$$

III. Si S est un simplexe, donc $S \subset R_n$, on a extr (S) = tend (S). Le tend (S) est composé des q points $(1 \leqslant q \leqslant n+1)$ constituant les sommets de S.

V. Soit $A \in \text{comp}(R_n)$. Le plus petit ensemble compact et convexe contenant A sera désigné par env (A).

V. Soient $a \in R_n$, $b \in R_n$, A et B non vides et bornés. $r(a, b)$ désigne la distance euclidienne de a et b. Nous posons $r(a, B) = \inf(r(a, b))$ pour $b \in B$. Posons $p = \sup(r(a, B))$ pour $a \in A$ et $q = \sup(r(b, A))$ pour $b \in B$. Le nombre $r^*(A, B) = \max(p, q)$ est dit distance au sens de Hausdorff entre les ensembles A et B. On n'envisagera dans la suite $r^*(A, B)$ que pour $A \in \text{comp}(R_n)$, $B \in \text{comp}(R_n)$. La famille R^* de tous les $A \in \text{comp}(R_n)$ constitue un espace métrique. $A_i \to A$ veut dire que $r^*(A_i, A) \to 0$. La suite $\{A_i\}$ est dite bornée lorsque $A_i \subset K$ où

$K \in \mathrm{comp}\,(R_n)$. Toute suite bornée contient une suite partielle convergente (cf. par ex. [2]). En ce sens R^* est un espace compact.

LEMME 1. *Si S est un simplexe, $a \in S \in R_n$,*

$$r(a, \mathrm{tend}\,(S)) \geqslant k > 0, \tag{2.1}$$

il existe un segment J de centre a et de longueur $2k/n$, tel que $J \subset S$.

Démonstration. Soient $s_1, \ldots, s_q$ les sommets de S et $m_1, \ldots, m_q$ — les coordonnées barycentriques de a. Étant donné que $s_i \neq a$, on aura $0 \leqslant m_i < 1$, $\sum m_i = 1$, $a = \sum m_i s_i$. Pour un certain indice j on aura $m_j \geqslant 1/q \geqslant 1/(n+1)$. La droite joignant a et s_j traverse la face de S opposée à s_j au point $u_j = a - p_j(s_j - a)$ où $p_j = m_j/(1 - m_j) \geqslant 1/(q-1) \geqslant 1/n$. Pour le segment fermé $[s_j, u_j]$ on aura $a \in [s_j, u_j] \subset S$ et $r(s_j, a) \geqslant k$, $r(u_j, a) \geqslant k/(q-1) \geqslant k/n$. Ce segment contient donc un segment J en question.

LEMME 2. *Si $F \in \mathrm{conv}$, $k > 0$*

$$a \in F,\ r(a, \mathrm{tend}\,(F)) > k, \tag{2.2}$$

alors il existe un segment J de centre a et de longueur $2k/n$ tel que $J \subset F$.

Démonstration. On a (cf. (2.2) et (1.1))

$$r(a, \mathrm{extr}\,(F)) > k. \tag{2.3}$$

En vertu de (2.2) il existe (cf. [1], p. 9) un simplexe S tel que $a \in S$, $\mathrm{extr}\,(S) \subset \mathrm{extr}\,(F)$, $S \subset F$. Il s'ensuit (cf. (2.3)) que $r(a, \mathrm{tend}\,(S)) = r(a, \mathrm{extr}\,(S)) > k$. En appliquant le Lemme 1, on obtient un segment J vérifiant aussi la thèse du présent Lemme,

LEMME 3. *Si $k > 0$, $H \in \mathrm{conv}$, $H_i \in \mathrm{conv}$, $a_i \in H_i$, $r(a_i, \mathrm{tend}\,(H_i)) \geqslant k$, $(i = 1, 2, \ldots)$, $H_i \to H$, $a_i \to a$, alors $a \in H$ et a n'est pas un point extrême de H.*

Démonstration. En vertu du Lemme 2, il existe un segment fermé $J_i \subset H_i$ de centre a_i et de longueur $2k/n$. Pour une suite partielle $\{J_{p(i)}\}$ de $\{J_i\}$ on aura $J_{p(i)} \to J$ où J est un segment de centre a, de longueur $2k/n$ et tel que $J \subset H$. Il s'ensuit que a n'est pas un point extrême de H.

DÉFINITION 1. Soit

$$A_i \in \mathrm{comp}\,(R_n),\ (i = 1, 2, \ldots),\ \{A_i\} \text{ est bornée.} \tag{3.1}$$

Nous avons par définition $\liminf A_i = \{b : b \in R_n,\ r(b, A_i) \to 0\}$.

Remarque 1. De l'hypothèse (3.1) suivent les propriétés

1° $\liminf A_i$ existe et est compact ou vide.

2° Soit C la famille des B compacts, tels que pour une suite partielle $A_{p(i)}$ on ait $A_{p(i)} \to B$ et soit M la classe des points appartenant a tous les ensembles B de la famille C. On a alors $\liminf A_i = M$.

3° La relation $A \subset \liminf A_i$ est équivalente à ce que l'on ait $A \subset \lim A_{p(i)}$ pour toute suite partielle convergente.

THÉORÈME 1. *Si* $E_i \in \mathrm{conv}$, $(i = 1, 2, \ldots)$

$$E_i \to E, \tag{3.2}$$

alors

$$\mathrm{tend}\,(E) \subset \liminf (\mathrm{tend}\,(E_i)). \tag{3.3}$$

Démonstration. Afin de prouver (3.3), il suffit de démontrer (cf. Rem. 1, 3°) que, si pour une suite partielle et pour un G compact on a

$$\mathrm{tend}\,(E_{p(i)}) \to G, \tag{3.4}$$

alors

$$\mathrm{tend}\,(E) \subset G. \tag{3.5}$$

Admettons que (3.5) n'a pas lieu. Il existe donc un $a \in \mathrm{tend}\,(E)$ tel que $r(a, G) > 0$ et, par conséquent (cf. (1.1)), il existe un b tel que $r(b, G) > 0$ et

$$b \in \mathrm{extr}\,(E) \subset E. \tag{3.6}$$

Pour un k fixe on a $r(b, G) = 3k > 0$, donc (cf. (3.4)) pour i grands on aura

$$r(b, \mathrm{tend}\,(E_{p(i)})) \geqslant 2k. \tag{3.7}$$

En vertu de (3.2) et (3.6) il existe un $b_{p(i)}$, tel que $b_{p(i)} \in E_{p(i)}$, $b_{p(i)} \to b$. Pour i grands on aura donc (cf. (3.7)) $r(b_{p(i)}, \mathrm{tend}\,(E_{p(i)})) \geqslant k$. Il s'ensuit, en vertu du Lemme 3, que b n'est pas un point extrême de E, ce qui est en contradiction avec (3.6).

PROPOSITION 1. *Si*

$$A \in \mathrm{comp}\,(R_n), \quad E = \mathrm{env}\,(A), \tag{4.1}$$

alors

$$\mathrm{tend}\,(\mathrm{env}\,(A)) \subset A, \tag{4.2}$$

$$\mathrm{env}\,(\mathrm{tend}\,(E)) = E. \tag{4.3}$$

Démonstration. Afin de prouver (4.2) il suffit de démontrer (cf. (4.1), (1.1)) que

$$\mathrm{extr}\,(\mathrm{env}\,(A)) \subset A. \tag{4.4}$$

Pour la démonstration par l'impossible admettons qu'il existe un p tel que

$$p \in \mathrm{extr}\,(\mathrm{env}\,(A)), \tag{4.5}$$

$$p \text{ n'appartient pas à } A. \tag{4.6}$$

On a (cf. (4.5)) $p \in \mathrm{env}\,(A)$. Il existe donc (cf. [1], p. 9) un simplexe S, tel que $p \in S$, $\mathrm{extr}\,(S) \subset A$, $S \subset \mathrm{env}\,(A)$. Il s'ensuit, en raison de (4.6), que p n'appartient pas à $\mathrm{extr}\,(S)$. Le point p appartient donc à l'interieur d'un segment $J \subset S \subset \mathrm{env}\,(A)$, ce qui est en contradiction avec (4.5).

Nous allons prouver maintenant (4.3). Vu que $\operatorname{env}(\operatorname{tend}(E)) \subset \operatorname{env}(E) = E$ il reste à prouver que $E \subset \operatorname{env}(\operatorname{tend}(E))$. Soit $p \in E$. Il existe (cf. [1], p. 15) un simplexe S, tel que $p \in S \subset E$, $\operatorname{extr}(S) \subset \operatorname{extr}(E) \subset \operatorname{tend}, (E)$, donc $p \in S = \operatorname{env}(\operatorname{extr}(S)) \subset \operatorname{env}(\operatorname{tend}(E))$, ce qui achève la démonstration.

Corollaire 1. *Si* $A \in \operatorname{comp}(R_n)$, $C = \operatorname{tend}(\operatorname{env}(A))$,

$$B \in \operatorname{comp}(R_n), \ \operatorname{env}(B) = \operatorname{env}(A), \tag{4.7}$$

alors $C \subset B$.

Définition 2. Pour $A \in \operatorname{comp}(R_n)$ nous désignons par $\operatorname{tend}(A)$ le plus petit ensemble B pour lequel (4.7) est vrai.

De cette définition, de la Proposition 1 et du Corollaire 1 résulte le

Théorème 2. 1°) *Au cas d'un A convexe et compact la formule* (1.1) *et la Définition 2 fournissent le même* $\operatorname{tend}(A)$. 2°) *Si* $A \in \operatorname{comp}(R_n)$, *la Définition 2 fournit un seul* $\operatorname{tend}(A)$ *et*

$$\operatorname{tend}(A) = \operatorname{tend}(\operatorname{env}(A)) \subset A \subset \operatorname{env}(A) = \operatorname{env}(\operatorname{tend}(A)).$$

3°) *Si* $A \in \operatorname{comp}(R_n)$, $B \in \operatorname{comp}(R_n)$, *la relation* $\operatorname{tend}(A) = \operatorname{tend}(B)$ *est eqivalente à la relation* $\operatorname{env}(A) = \operatorname{env}(B)$.

Théorème 3. *Si* $A_i \in \operatorname{comp}(R_n)$, $(i = 1, 2, \ldots)$, $A \in \operatorname{comp}(R_n)$ *et si pour* $i \to \infty$ *on a* $A_i \to A$ (*au sens de Hausdorff*) *alors* $\operatorname{tend}(A) \subset \liminf(\operatorname{tend}(A_i))$.

Démonstration. Posons $E = \operatorname{env}(A)$, $E_i = \operatorname{env}(A_i)$. On prouve que $E_i \to E$. Il devient ainsi clair que le présent Théorème résulte des Théorèmes 1 et 2.

Définition 3. Soit $t \in T \subset R_1$ et soit $B(t) \in \operatorname{comp}(R_n)$ pour $t \in T$. Si des relations $t_i \in T$, $t_0 \in T$, $t_i \to t_0$ il résulte que $B(t_0) \subset \liminf(B(t_i))$, alors on dit que $B(t)$ est sémicontinue inférieurement pour $t = t_0$.

Remarque 2. Soit $p > 0$ et soit $V(B, p)$ le voisinage fermé de rayon p de l'ensemble $B \subset R_n$. La semicontinuité de $B(t)$ pour $t = t_0$ revient à ce qu'à tout $p > 0$ correspond un $q > 0$ tel que $B(t_0) \subset V(B(t), p)$ lorsque $t \in T$ et $|t - t_0| \leqslant q$.

Le Théorème 3 donne le

Théorème 4. *Si* $A(t) \in \operatorname{comp}(R_n)$ *et* $A(t)$ *est continue au sens de Hausdorff dans* $T \subset R_1$, *alors* $\operatorname{tend}(A(t))$ *est semicontinue inférieurement dans T.*

INSTITUT MATHÉMATIQUE, ACADÉMIE POLONAISE DES SCIENCES
(INSTYTUT MATEMATYCZNY, PAN)

OUVRAGES CITÉS

[1] T. Bonnesen und W. Fenchel, *Theorie der konvexen Körper*, New York, (1948)
[2] T. Ważewski, *Sur un continu singulier*, Fund. Math., **4** (1923).

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 12, 1961

MATHEMATICS

Concerning the Deflation Principle for Linear Iterations

by

M. ALTMAN

Presented by W. ORLICZ on October 24, 1961

1. Paper [1] contains a deflation principle for linear iteration procedures. The idea of this principle consists in introducing an auxiliary subspace in such a manner that the iteration process is carried out in some complementary subspace which is there called the iteration carrier. Thus, this principle gives a method of embedding the iteration process in some subspace. The purpose of such embedding is to improve the condition number of the operator, since the speed of the convergence of many iterative methods depends on this number. Thus, by a suitable choice of the auxiliary subspace we get a method of speeding-up the convergence of the iterative process.

This paper contains another deflation principle for linear iteration procedures. We use here the same concept as in [1]. However, there is a difference between these methods. The principle in [1] gives a transformation which reduces the original equation to a homogeneous one, while the principle defined here gives also such transformation, but the equation reduced in this way is non-homogeneous. Besides, the iteration carrier subspace introduced in [1] is always orthogonal to vector b of Eq. (1), while the same subspace defined in the principle considered here does not require this property and can be an arbitrary subspace. In addition, if the auxiliary subspaces in both principles are the same, the deflation index defined in [1] is by one greater than the same index in the principle given here.

Let A be a linear (i.e. additive and homogeneous) operator with domain and range in a Hilbert space H. Consider the linear equation

$$Ax = b, \qquad x, b \in H. \tag{1}$$

Let us assume that operator A is self-adjoint and positive definite, or, more precisely,

$$A^* = A,$$

where A^* is the adjoint of A, and

$$m(x, x) \leqslant (Ax, x) \leqslant M(x, x),$$

where $0 < m < M$, m and M being the minimum and maximum eigenvalues of A, respectively.

Let L be an arbitrary subspace of H. Let us put

$$Ł = L(A),$$

i.e. $Ł$ is the image of the subspace L. Let Z be the complementary subspace of H orthogonal to $Ł$, i.e.

$$Z \perp Ł, \quad Ł, Z \subset H.$$

Denote by $P_Ł x$, $P_Z x$ the orthogonal projection of x onto the subspace $Ł$, Z of H, respectively.

We have

$$P_Z x = x - P_Ł x. \tag{2}$$

Let us put

$$b' = b - P_Ł b.$$

Hence, we get

$$b' = b - A\tilde{b}, \tag{3}$$

where, by definition,

$$P_Ł b = A\tilde{b}.$$

Instead of Eq. (1) consider now the following equation

$$Ax = b'. \tag{4}$$

Thus, if x^* is the solution of Eq. (1) and x' is the solution of Eq. (4), then we have, by (3),

$$x^* = x' + \tilde{b}. \tag{5}$$

Let us now define the linear transformation R as follows

$$z = Rx = P_Z Ax = Ax - P_Ł Ax, \tag{6}$$

in virtue of (2).

Let us put

$$A\tilde{x} = P_Ł Ax. \tag{7}$$

Thus, we get, by (6) and (7),

$$z = Rx = Ax - A\tilde{x}. \tag{8}$$

Consider now the following linear equation

$$Rx = b', \tag{9}$$

where operator R is defined by (8)

It is easy to see that the solution of Eq. (1) can be reduced to that of Eq. (9). In fact, if x^* is the solution of Eq. (1), then we have, by (4) and (8),

$$x^* = x - \tilde{x} + \tilde{b},$$

where x is the solution of Eq. (9).

Let us now consider operator R defined only on the subspace Z. It is easy to verify that the following relation is valid

$$(Rz, z) = (Az, z) \tag{10}$$

for any z of Z. Hence, it follows that operator R is positive definite and self-adjoint on the whole of Z. Denote by m_Z, M_Z the minimum and maximum eigenvalues of R in Z, respectively. Thus, we obtain, by (10),

$$m \leqslant m_Z, \quad M_Z \leqslant M, \tag{11}$$

and for any z of Z we get

$$m_Z(z, z) \leqslant (Rz, z) \leqslant M_Z(z, z). \tag{12}$$

Relations (10)—(12) show that by a suitable choice of the auxiliary subspace L the condition number of operator R can be much better than that of operator A. The speed of many iterative methods depends actually on the condition number. Hence, it seems useful to apply these methods to the solution of Eq. (9) instead of Eq. (1). But this is equivalent to the embedding of the iteration process in the subspace Z. This embedding is accomplished by the transformation R. In other words, we obtain in this way a general deflation principle for linear iteration procedures. The deflation principle defined by the transformation R gives actually a method of speeding-up the convergence. The subspace L is the auxiliary subspace mentioned above.

If the subspace L consists only of the zero element, then we obtain, of course, the original iteration process without change.

2. Applying the deflation principle defined in par. 1 to different iterative methods we obtain various methods with deflation. We shall now consider gradient methods with deflation.

Let x_0 be an arbitrary element of Z. Put

$$x_{n+1} = x_n - a_n r_n, \quad n = 0, 1, 2, \ldots, \tag{13}$$

where

$$r_n = b' - Rx_n.$$

Hence it follows that

$$r_{n+1} = r_n - a_n Rr_n. \tag{14}$$

The approximate solution y_n of Eq. (1) is then defined by the following formula

$$y_n = x_n - \tilde{x}_n + \tilde{b}, \tag{15}$$

in virtue of (3) and (8).

Formulae (13), (14) define a general gradient method with deation.fl If the coefficients a_n in relations (13), (14) are defined by the following formula

$$a_n = \frac{(r_n, r_n)}{(Rr_n, r_n)},$$

then we obtain the method of steepest descent [2] with deflation.

If the coefficients a_n in relations (13), (14) are defined by the following formula

$$a_n = \frac{(Rr_n, r_n)}{\|Rr_n\|^2},$$

then we obtain the minimum residua method [3] with deflation.

In both cases the sequence of approximate solutions x_n of Eq. (4) defined by (13) and (14) converges to the solution of Eq. (4) at least as fast as a geometric progression with ratio

$$q_Z = \frac{M_Z - m_Z}{M_Z + m_Z}. \tag{16}$$

It results from (3) and (8) that

$$r_n = b' - Rx_n = b - A\tilde{b} - A(x_n - \tilde{x}_n).$$

Thus, in virtue of (15), we get

$$r_n = b - Ay_n. \tag{17}$$

Hence, it follows that the sequence of approximate solutions y_n of Eq. (1) converges to the solution of Eq. (1) at least as fast as a geometric progression with the same ratio q_Z defined by (16).

In virtue of (17) we get the following error estimate for the approximate solution y_n of Eq. (1)

$$\|x^* - y_n\| \leqslant \frac{\|r_0\|}{m} q_Z^n,$$

where x^* it the solution of Eq. (1).

Remark 1. Let v_0 be an arbitrary element of H and put

$$x_0 = v_0 - P_{\mathcal{L}} v_0.$$

Then $x_0 \in Z$ and one can start with x_0 to get the approximate solution x_n defined by (13) and (14).

Remark 2. Suppose that operator A is defined by the matrix.

$$A = (a_{ik}), \quad i, k = 1, 2, \ldots, N.$$

Let $l_1, l_2, \ldots, l_s$ be linearly independent vectors. Orthogonalizing $Al_1, Al_2, \ldots, Al_s$ we obtain an orthonormal system $u_1, u_2, \ldots, u_s$, $(s < N)$ forming a basis in the subspace $\mathcal{L}$. Then operator R can be obtained in a matrix form as follows. Put

$$b = (b_1, b_2, \ldots, b_N), \quad a_k = (a_{k1}, a_{2k}, \ldots, a_{Nk}),$$
$$u_j = (u_{1j}, u_{2j}, \ldots, u_{Nj}), \quad d_{kj} = (a_k, u_j), \quad j = 1, 2, \ldots, s.$$

We get thus the following matrix for the operator R

$$R = (r_{ik}) \quad i, k = 1, 2, \ldots, N,$$

where $r_{ik} = a_{ik} - d_{k1} u_{i1} - d_{k2} u_{i2} - \ldots - d_{ks} u_{is}$.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY, PAN)

REFERENCES

[1] M. Altman, *A general deflation principle for linear iterations*, Bull. Acad. Polon. Sci., Sér. sci. math. astr. et phys., **9** (1961), 000.

[2] L. V. Kantorovitch, *Functional analysis and applied mathematics* [in Russian], Usp. Mat. Nauk (NS), **3** (1948), 89—185, [transl. into English: Nat. Bureau of Stand.].

[3] M. A. Krasnoselsky and S. G. Krein, *Iteration process with minimum residua* [in Russian], Mat. Sbornik (NS), **31** (1952), 315—334.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 12, 1961

MATHEMATICS

Connection between Gradient Methods and Newton's Method for Functionals

by

M. ALTMAN

Presented by W. ORLICZ on October 24, 1961

1. In [1] we presented a generalization of Newton's well-known classical method for solving non-linear functional equations in Banach spaces, more precisely, for finding the zero elements of non-linear functionals in such spaces. This method can be applied to solve linear and non-linear operator equations in Banach spaces. Our generalization is essentially different from that given by L. V. Kantorovitch [4]. Moreover, the difficulties of applying the Newton—Kantorovitch method in practice are very great. We have there such hypotheses like the existence of the inverse of the Fréchet derivative and the estimate of its norm. In addition, using Kantorovitch's method we have to solve a linear operator equation at each iteration step. All these difficulties are obviated in our method.

In [2] we investigated the connection between the method of steepest descent and our abstract variant of Newton's method. In this paper we examine the connection between our method and other gradient methods of solving linear equations in Hilbert space. It appears that not only the method of steepest descent is a particular case of Newton's method for non-linear functionals. The same is also true for other gradient methods, for instance, for the method with minimum residua investigated by M. A. Krasnoselsky and S. G. Krein [6]. So called gradient methods with relaxation can also be mentioned here. The difference between these various gradient methods consists in the choice of a suitable non-linear functional and a Hilbert space. Thus, we obtain different gradient methods by choosing different functionals and Hilbert spaces.

Let X be a Banach space and let $F(x)$, $x \in X$, be a non-linear continuous functional defined on some closed solid sphere $S(x_0, r)$ of X with centre x_0 and radius r.

Consider the non-linear functional equation

$$F(x) = 0. \tag{1}$$

Let us assume that $F(x)$ is differentiable in Fréchet's sense in the sphere $S(x_0, r)$. Then the iteration process of solving Eq. (1) is defined by the following formula

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)\, y_n} y_n, \qquad n = 0, 1, 2, \ldots, \tag{2}$$

where x_0 is the initial approximate solution and the elements y_n are so chosen as to satisfy the following condition

$$\|y_n\| = 1 \quad F'(x_n)\, y_n = \|F'(x_n)\|, \tag{3}$$

provided that such a choice is possible. In the case when condition (3) cannot be satisfied, it is enough to choose an element y_n such that the value $F'(x_n)\, y_n$ is sufficiently near to the norm $\|F'(x_n)\|$ (see [3]).

Let A be a linear (i.e. additive and homogeneous) operator with domain and range in a Hilbert space H. Weigh now the linear equation

$$Ax = b, \quad x, b \in H. \tag{4}$$

We shall assume operator A to be positive definite and self-adjoint, or, more precisely,

$$m(x, x) \leqslant (Ax, x) \leqslant M(x, x),$$

where $0 < m < M < \infty$, m, M being the minimum and maximum eigenvalues of A, respectively, and $A^* = A$, where A^* is the adjoint of A.

There are many ways of reducing Eq. (4) to Eq. (1). In fact, putting

$$F(x) = \|Ax - b\| \tag{5}$$

and introducing a new scalar product in H

$$[u, v] = (Au, v), \quad u, v \in H, \tag{6}$$

we get a new Hilbert space H_A and a non-linear functional defined on H_A by formula (5).

Let us remark that if X is a Hilbert space, then process (2) becomes of the following form

$$x_{n+1} = x_n - \frac{F(x_n)}{\|F'(x_n)\|^2} F'(x_n). \tag{7}$$

In virtue of (7), we obtain that Newton's method for the functional defined by (5) on the Hilbert space H_A gives the following iteration process

$$x_{n+1} = x_n - \frac{\|Ax - b\|^2}{(A(Ax_n - b), Ax_n - b)} (Ax_n - b). \tag{8}$$

But formula (8) determines exactly the method of steepest descent [4].

2. Another way of reducing Eq. (4) to Eq. (1) can be given as follows. Put

$$F(x) = (A(Ax - b), Ax - b)^{1/2}, \tag{9}$$

and introduce a new scalar product in the space H, using the following formula

$$[u, v] = (A^2 u, v), \quad u, v \in H. \tag{10}$$

Thus, we obtain a non-linear functional defined on the Hilbert space H_{A^2} by formula (9). It is easy to see that process (7) yields here

$$x_{n+1} = x_n - \frac{(A(Ax_n - b).\, Ax_n - b)}{\|A(Ax_n - b)\|^2} (Ax_n - b). \tag{11}$$

But formula (11) is exactly the same as in the minimum residua method [6]. Hence, it follows that the method with minimum residua is a particular case of the generalized Newton's method for functionals.

3. Weigh now the following equation

$$F^{\alpha}(x) = 0 \tag{12}$$

instead of Eq. (1), α beig a real number. A simple calculation shows that process (7) for Eq. (12) becomes of the following form

$$x_{n+1} = x_n - \frac{F(x_n)}{\alpha \| F'(x_n) \|^2} F'(x_n). \tag{13}$$

Thus, we see that one can introduce a parameter α in the process (7) by app'ying the same process (7) to Eq. (12) instead of Eq. (1). These facts are equivalent.

As an application of this observation we obtain the normal chord method investigated by Kostarchuk [5]. Thus, replacing the functional in (5) by

$$F(x) = \| Ax - b \|^{1/2}$$

defined on H_A we get the following iteration process instead of (8)

$$x_{n+1} = x_n - 2 \frac{\| Ax_n - b \|^2}{(A(Ax_n - b),\ Ax_n - b)} (Ax_n - b). \tag{14}$$

Formula (14) defines the normal chord method mentioned above.

4. We shall now consider the general case, when A is a non-singular linear bounded operator. To apply the method defined by formula (7) we put $F(x)$ defined on H by (5). Then we get the following iteration process

$$x_{n+1} = x_n - \frac{\| Ax_n - b \|^2}{\| A^*(Ax_n - b) \|^2} A^*(Ax_n - b). \tag{15}$$

We shall prove that the sequence of the approximate solutions x_n defined by (15) converges to the solution x^* of Eq. (4). We have

$$\| x_{n+1} - x^* \|^2 = \| x_n - x^* \|^2 - \frac{\| Ax_n - b \|^4}{\| A^*(Ax_n - b) \|^2}. \tag{16}$$

Formula (16) shows the convergence of the sequence of $x_n - x^*$. In fact, this sequence is decreasing and non-negative. Since the denominator in (16) is bounded, it follows from (16) that sequence of $\| Ax_n - b \|^4$ converges to zero.

Introducing a parameter α into (15) we get the following iteration process

$$x_{n+1} = x_n - \alpha \frac{A \| Ax_n - b \|^2}{\| A^*(Ax_n - b) \|^2} A^*(Ax_n - b). \tag{17}$$

Then we obtain the following relation instead of (16)

$$\|x_{n+1} - x^*\|^2 = \|x_n - x^*\|^2 - \alpha(2 - \alpha)\frac{\|Ax_n - b\|^4}{\|A^*(Ax_n - b)\|^2}. \tag{18}$$

Let us assume that

$$0 < \alpha < 2.$$

If this condition is satisfied, then we can apply the same argument as before and we get the convergence of x_n defined by (17) to the solution x^* of Eq. (4).

Let us remark that $\|x_{n+1} - x^*\|^2$ is minimized for $\alpha = 1$. This fact follows immediately from relation (18).

INSTITUTE OF MATHEMATICS POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY, PAN)

REFERENCES

[1] M. Altman, *Concerning approximate solutions of non-linear functional equations*, Bull. Acad. Polon. Sci., Cl. III, 5 (1957), 461—465.

[2] — , *Connection between the method of steepest descent and Newton's method*, ibid., 1031—1036.

[3] — , *Extension and stability of certain iterative methods for solving non-linear functional equations in Babach spaces*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **9** (1961), 267—271.

[4] L. V. Kantorovitch, *Functional analysis and applied mathematics* [in Russian], Usp. Mat. Nauk, (NS) 3, (1948) No. 6, (28), 89—185 [transl. into English: Nat. Bureau of Stand.).

[5] V. N. Kostarchuk, *The normal chord method of solving linear operator equations* [in Russian], Trudy Semin. po funkzional. anal., Voronež, Nos. 3—4 (1960), 54—76.

[6] M. A. Krasnoselsky and S. G. Krein, *Iteration process with minimum residua* [in Russian], Matem. Sbornik (NS), **31** (1952), 315—334.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 12, 1961

MATHEMATICS

Set Existence II

by

R. O. GANDY, G. KREISEL and W. W. TAIT

Presented by A. MOSTOWSKI on October 28, 1961

The present note contains some minor additions and corrections to our paper [2].

1. Theorem 1 of [2], p. 580, may be improved as follows.

The conclusion of Theorem 2 (i) holds under the hypothesis that the set of axioms $\mathcal{A}$ contains the successor axioms of paragraph 1 (without additional assumptions).

For, either $\mathcal{A}$ has no model in which distinct numerals denote distinct objects, in which case the conclusion holds trivially. Or else we may consistently assume that each model of $\mathcal{A}$ can be partially ordered (i.e. add a new relation symbol $<$ with the axiom below) in such a way that the denotations of the numerals come first in this order. This is expressed by the set of axioms:

$$(x) \neg (x < 0)$$

$$(x)(x_1) \ldots (x_{n+1}) \{[S(0, x_1) \,\&\, S(x_1, x_2) \,\&\, \ldots \,\&\, S(x_n, x_{n+1})] \rightarrow .$$

$$(x < x_{n+1} \vee x = x_{n+1} \vee x_{n+1} < x) \wedge [x_{n+1} \leftrightarrow (x = 0 \vee x = x_1 \vee \ldots \vee x = x_n)]\}.$$

(These axioms suffice for the proof of (1) in the text).

The first sentence of the proof of Theorem 2 (ii), p. 580, lines 2 to 10 from the bottom, should read:

Note first that if H_n occurs in an ω-model, it is there uniquely defined by a suitable arithmetic condition $\mathfrak{Z}(X^1, n)$. So either H_n does not occur in some ω-model of $\mathcal{A}$, when, by the assumed form of the comprehension axiom, no H_m, $|m| \geqslant |n|$ is assured either, and so Theorem 2 (ii) is proved. Or else we proceed as follows.

2. Theorem 2 (ii) has since been used in [4], p. 384, par. 3 to sharpen the result on the impredicativity of the theorem of Cantor—Bendixon established in [3].

3. Corrections. p. 578. The referee has informed us that his proof of the "refinement" (mentioned in line 3 of the Remark on p. 578) was not correct. It is hoped that he will publish a definitive version.

p. 578, line 10: read "basis" for "basic".

p. 580, footnote: delete "an". Add: Gandy's construction has appeared in [1].

p. 582, replace lines 4 and 3 from bottom by

[2] R. O. Gandy, *On a problem of Kleene's*, Bull. Amer. Math. Soc., 66 (1960), 501—502.

[3] C. Kreisel, *Set theoretic problems suggested by the notion of potential totalities*, Proceedings of the symposium on infinitistic methods in the foundations of mathematics, Warsaw 2.9—8.9 1959 (1961), p. 103—140.

UNIVERSITY OF LEEDS (ENGLAND)
UNIVERSITY OF PARIS (FRANCE)
STANFORD UNIVERSITY, CALIFORNIA (USA)

REFERENCES

[1] R. O. Gandy, *Proof of Mostowski's conjecture*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **8** (1960), 571—575.

[2] R. O. Gandy, G. Kreisel and W. W. Tait, *Set existence*, ibid., **8** (1960), 576—582.

[3] G. Kreisel, *Analysis of the Cantor—Bendixon theorem by means of the analytic hierarchy*, ibid., **7** (1959), 621—626.

[4] G. Kreisel, *La prédicativité*, Bull. Soc. Math., de France, **88** (1960), 371—391.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 12, 1961

THEORETICAL PHYSICS

Selection Rules for Electric Dipole Transitions in Rutile

by

K. OLBRYCHSKI

Presented by L. INFELD on October 14, 1961

In the preceding paper [1] were given the representations for symmetry group of rutile. Now we shall use them to give selection rules for optical transitions (electric dipole).

I. General theory

1. Introduction. For system weakly interacting with electromagnetic field the probability amplitude of electric dipole transition per unit of time from state $|\psi_i\rangle$ to state $|\psi_f\rangle$ is proportional to the matrix element:

$$M_{if} \equiv \langle \psi_i \, | \boldsymbol{n} \operatorname{grad} | \, \psi_f \rangle, \tag{1}$$

where $\boldsymbol{n}$ is the versor parallel to the electric vector of light [2]. We shall find by means of group theory the cases where this matrix element vanishes. We take for ψ_i and ψ_f the two-component one-electron wave functions (Pauli spinors); this contains also the case of one-component wave functions.

2. The transformation properties of grad ψ_f. Let us denote the space (double) group of symmetry of crystal by G. The acting of an element $A = [\alpha|t]$ of G on the spinor $\varphi(\boldsymbol{r})$ is defined as follows:

$$A \cdot \varphi(\boldsymbol{r}) = D_{1/2}(\alpha)\, \varphi(A^{-1}\, \boldsymbol{r}),$$

where $D_{1/2}(\alpha)$ are well known spinor matrices defined as in [1]. The spinors ψ_i and ψ_f are eigenfunctions of the unperturbed Hamiltonian, therefore they transform under elements A according to irreducible representations of G, D^i and D^f, respectively.

The operators $a(A)$ are elements of point symmetry group of crystal; they form simultaneously a 3-dimensional representation of G, reducible in general, corresponding to the point $\Gamma(\boldsymbol{k} = 0)$ of Brillouin Zone.

Because

$$\operatorname{grad} \psi_f(A^{-1}\, \boldsymbol{r}) = a(A)\, [\operatorname{grad} \psi_f(\boldsymbol{r})]_{\boldsymbol{r} \to A^{-1}\, \boldsymbol{r}}$$

therefore, multiplying on the left by $a^{-1}(A)$ and $D_{1/2}(\alpha)$ we have:

$$A \cdot \operatorname{grad} \psi_f \equiv D_{1/2}(\alpha)\, [\operatorname{grad} \psi_f(\boldsymbol{r})]_{\boldsymbol{r} \to A^{-1}\, \boldsymbol{r}} = a^{-1}(A) \operatorname{grad} D_{1/2}(\alpha)\, \psi_f(A^{-1}\, \boldsymbol{r}). \tag{2}$$

We see, that each component of gradient transforms according to a representation being the Kronecker product of the representation $\alpha(A)$ acting in space of operators $\partial/\partial x_j$ ($j = 1, 2, 3$) by D^f (we use the fact, that the matrices $\alpha(A)$ are orthogonal).

3. LEMMA. *Let φ_q ($q = 1, ..., d$) be basis functions for irreducible representation D^s of a given group F. We assume that we can expand each of them into a series of linear independent functions ψ_{nm} generating irreducible representations of F, D^n respectively*:

$$\varphi_q = \sum_{nm} a_{q,\,nm}\,\psi_{nm}.$$

We assume here without loss of generality that the equivalent representations from among the considered ones are identical. In general, representations D^n having different n can be identical. Then:

a) in this expansion will enter only those ψ_{nm} which generate representations D^n identical with D^s,

b) the coefficients $a_{q,\,nm}$ are independent of q, and different from zero only for $m = q$.

In the proof [3] one uses the orthogonality theorem for irreducible unitary representations:

$$\frac{1}{N}\sum_{g\in F} D^{s*}_{pq}(g)\,D^{s'}_{p'q'}(g) = \frac{1}{\dim D^s}\,\delta_{ss'}\,\delta_{pp'}\,\delta_{qq'} \qquad (N \text{ — order of } F).$$

4. Deriving of the selection rules. To apply the above Lemma in our case let us remark that each term of the sum

$$\sum_{j=1,2,3} n_j \frac{\partial}{\partial x_j}\,\psi_f \tag{3}$$

transforms under elements of G according to representation $\alpha\times D^f$ reducible in general. We can exhibit each term of this sum as the linear combination of basis functions for irreducible representations, into which $\alpha\times D^f$ is decomposable.

If in our Lemma we take for φ_q each of these basis functions, for F — the symmetry group of crystal G, for ψ_{nm} — the eigenfunctions of our unperturbed Hamiltonian, where

$$\langle\psi_{nm}|\psi_{n'm'}\rangle = \delta_{nn'}\,\delta_{mm'}, \tag{4}$$

then all conditions assumed in the Lemma will be fulfilled. Therefore, in expansion of expression (3) into a series of Hamiltonian eigenfunctions will appear only those ψ_{nm} which generate the irreducible representations of G into which $\alpha\times D^f$ is decomposable. From (1) and (4) we see that transitions from the state $|\psi_f\rangle$ are possible only to the states $|\psi_i\rangle$ for which D^i is one from the representations, direct sum of which is $\alpha\times D^f$ (because the operator $\boldsymbol{n}$ grad is antihermitean, the probabilities of transitions $i\to f$ and $f\to i$ are identical).

5. Simplifications. We assume that ψ_f has a definite value of $\boldsymbol{k}$, i.e. it is multiplied by $\exp(i\boldsymbol{k}\boldsymbol{R}_n)$ under translation through $\boldsymbol{R}_n$. It is easily seen that then each

term of the sum (3) also has this property. Therefore, under elements B of group $G(\boldsymbol{k})$ of wave vector $\boldsymbol{k}$ the function ψ_f transforms according to irreducible representation $D^f_{\boldsymbol{k}}(B)$, functions $\partial/\partial x_j\ \psi_f$ $(j = 1, 2, 3)$ — according to reducible in general representation $\alpha(B)\times D^f_{\boldsymbol{k}}(B)$. Using the transformation properties of the sum (3) under elements of $G(\boldsymbol{k})$, similarly as we did it in par. 4 for the group G, we shall get certain selection rules. These selection rules are equivalent to the selection rules of par. 4, because representations $D^f_{\boldsymbol{k}}(B)$ and $\alpha(B)\times D^f_{\boldsymbol{k}}(B)$ uniquely determine the representations $D^f(A)$ and $\alpha(A)\times D^f(A)$, respectively, and conversely (this follows from the general theory of representations of space groups).

6. Dependence of selection rules on $\boldsymbol{n}$. As follows from par. 5, to give the selection rules we can confine ourselves to examining of corresponding represen tations of $G(\boldsymbol{k})$. The matrices $\alpha^{-1}(B)$ from formula (2) are reducible in general, i.e. they are decomposable into irreducible matrices $(\alpha^b)^{-1}(B)$ which act e.g. in subspaces C^b of 3-dimensional complex space. Let us exhibit the operator grad as the sum:

$$\sum_{bl} (\boldsymbol{e}_{bl} \cdot \operatorname{grad})\, \boldsymbol{e}_{bl}\,,$$

where $\boldsymbol{e}_{bl}$ are versors spanning C^b. The bl-th component of gradient will be transform under elements of $G(\boldsymbol{k})$ according to the representation $\alpha^b(B)$. Therefore, if we choose the vector $\boldsymbol{n}$ in (1) lying in the subspace C^b, the expression (3) will be decomposab'e into a sum of functions transforming under $G(\boldsymbol{k})$ according to the representation $\alpha^b(B)\times D^f_{\boldsymbol{k}}(B)$. With the help of lemma we get the required selection rules dependent in general on the polarization direction of light.

For example, in case of the simple tetragonal lattice (as in rutile) the matrices $\alpha(A)$ have for all A the quasidiagonal form, hence for every point of Brillouin Zone we have at least two invariant subspaces C^b; this distinguishes the transitions with $\boldsymbol{n}$ perpendicular to symmetry axis of crystal (so-called σ-transitions [4]) as well as parallel (π-transitions).

7. Final remark. If we assume additionally that ψ_f transforms according to irreducible representation of some subgroup of G, then we can get additional selection rules. For instance, if we take for F in Lemma the subgroup of primitive translations (the function ψ_f as well as (3) are multiplied by $\exp(i\boldsymbol{k}\boldsymbol{R}_n)$ under $[\varepsilon|\boldsymbol{R}_n]$), we get the rule allowing only vertical transitions (this rule follows exceptionally also from considerations of par. 5). For similar purposes one can use also the point b) of Lemma. We shall not go into detail here.

II. Selection rules for rutile

For the points belonging to the interior of Brillouin Zone we use tables of characters for corresponding groups $G(\boldsymbol{k})$ given in [5]. In our tables we write the representations in the same order as Koster, but we sue different symbols for them. For the points on the surface of Brillouin Zone we use tables of characters for representations with weight, given in the preceding paper [1].

In the tables the symbol σ designates a transition with $\boldsymbol{n}$ perpendicular to vector (0, 0, 1), π — with $\boldsymbol{n} = (0, 0, 1)$, σ_1 — with $\boldsymbol{n} = (1, 0, 0)$, σ_2 — with $\boldsymbol{n} = 0, 1, 0)$, σ_1' — with $\boldsymbol{n} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, σ_2' — with $\boldsymbol{n} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$.

TABLE I

The point Γ

Initial state	Final states	
	σ	π
Γ_1	Γ_{10}	Γ_7
Γ_2	Γ_{10}	Γ_6
Γ_3	Γ_{10}	Γ_9
Γ_4	Γ_{10}	Γ_8
Γ_5	$\Gamma_6, \Gamma_7, \Gamma_8, \Gamma_9$	Γ_{10}
Γ_6	Γ_5	Γ_2
Γ_7	Γ_5	Γ_1
Γ_8	Γ_5	Γ_4
Γ_9	Γ_5	Γ_3
Γ_{10}	$\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$	Γ_5
Γ_{11}	Γ_{13}, Γ_{14}	Γ_{13}
Γ_{12}	Γ_{13}, Γ_{14}	Γ_{14}
Γ_{13}	Γ_{11}, Γ_{12}	Γ_{11}
Γ_{14}	Γ_{11}, Γ_{12}	Γ_{12}

TABLE II

The plane $\Delta\Sigma$

Initial state	Final states		
	σ_1	σ_2	π
$\Delta\Sigma_1$	$\Delta\Sigma_1$	$\Delta\Sigma_1$	$\Delta\Sigma_2$
$\Delta\Sigma_2$	$\Delta\Sigma_2$	$\Delta\Sigma_2$	$\Delta\Sigma_1$
$\Delta\Sigma_3$	$\Delta\Sigma_3$	$\Delta\Sigma_3$	$\Delta\Sigma_4$
$\Delta\Sigma_4$	$\Delta\Sigma_4$	$\Delta\Sigma_4$	$\Delta\Sigma_3$

From Table II we obtain the Table for the plane $\Delta\Delta$ by substituting:

$$\Delta\Sigma \to \Delta\Delta, \quad \sigma_1 \leftrightarrow \pi,$$

and for the plane $\Lambda\Sigma$ by substituting:

$$\Delta\Sigma \to \Lambda\Sigma, \quad \sigma_1 \to \sigma_1', \quad \sigma_2 \to \pi, \quad \pi \to \sigma_2'.$$

TABLE III

The axis Λ

Initial state	Final states	
	σ	π
Λ_1	Λ_5	Λ_1
Λ_2	Λ_5	Λ_2
Λ_3	Λ_5	Λ_3
Λ_4	Λ_5	Λ_4
Λ_5	$\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$	Λ_5
Λ_6	Λ_6, Λ_7	Λ_6
Λ_7	Λ_6, Λ_7	Λ_7

TABLE IV

The axis Δ

Initial state	Final states		
	σ_1	σ_2	π
Δ_1	Δ_2	Δ_1	Δ_4
Δ_2	Δ_1	Δ_2	Δ_3
Δ_3	Δ_4	Δ_3	Δ_2
Δ_4	Δ_3	Δ_4	Δ_1
Δ_5	Δ_5	Δ_5	Δ_5

From Table IV we obtain the Table for the axis Σ by substituting:

$$\Delta \to \Sigma, \quad \sigma_1 \to \sigma_2', \quad \sigma_2 \to \sigma_1'.$$

TABLE V

The plane US

Initial state	Final states		
	σ_1	σ_2	π
US_1	US_1	US_1	US_2
US_2	US_2	US_2	US_1
US_3	US_3	US_3	US_4
US_4	US_4	US_4	US_3

From Table V we obtain the Table for the plane VW by substituting:

$$US \to VW,\ \sigma_2 \leftrightarrow \pi.$$

TABLE VI

The axis Y

Initial state	Final states		
	σ_1	σ_2	π
Y_1	Y_1	Y_2	Y_4
Y_2	Y_2	Y_1	Y_3
Y_3	Y_3	Y_4	Y_2
Y_4	Y_4	Y_3	Y_1
Y_5	Y_5	Y_5	Y_5

From Table VI we obtain the Table for the axis S by substituting:

$$Y \to S,\ \sigma_1 \to \sigma_1',\ \sigma_2 \to \pi,\ \pi \to \sigma_2'.$$

TABLE VII

The axis T

Initial state	Final states		
	σ_1	σ_2	π
T_1	T_1	T_1	T_1
T_2	T_2	T_3	T_5
T_3	T_3	T_2	T_4
T_4	T_4	T_5	T_3
T_5	T_5	T_4	T_2

From Table VII we obtain the Table for the axis U by substituting:

$$T \to U,\ \sigma_1 \leftrightarrow \sigma_2$$

TABLE VIII

The axis W

Initial state	Final states		
	σ_1	σ_2	π
W_1	W_1	W_1	W_1
W_2	W_5	W_4	W_2
W_3	W_4	W_5	W_3
W_4	W_3	W_2	W_4
W_5	W_2	W_3	W_5

TABLE IX

The axis V

Initial state	Final states	
	σ	π
V_1	V_5	V_1
V_2	V_5	V_2
V_3	V_5	V_3
V_4	V_5	V_4
V_5	V_1, V_2, V_3, V_4	V_5
V_6	V_6, V_7	V_6
V_7	V_6, V_7	V_7

TABLE X

The point X

Initial state	Final states		
	σ_1	σ_2	π
X_1	X_1	X_1	X_2
X_2	X_2	X_2	X_1
X_3	X_4	X_4	X_3
X_4	X_3	X_3	X_4

TABLE XI

The point R

Initial state	Final states		
	σ_1	σ_2	π
R_1	R_2	R_2	R_2
R_2	R_1	R_1	R_1
R_3	R_8	R_{10}	R_6
R_4	R_7	R_9	R_5
R_5	R_{10}	R_8	R_4
R_6	R_9	R_7	R_3
R_7	R_4	R_6	R_{10}
R_8	R_3	R_5	R_9
R_9	R_6	R_4	R_8
R_{10}	R_5	R_3	R_7

TABLE XII

The point M

Initial state	Final states	
	σ	π
M_1	M_{10}	M_4
M_2	M_{10}	M_3
M_3	M_9	M_2
M_4	M_9	M_1
M_5	M_{10}	M_8
M_6	M_{10}	M_7
M_7	M_9	M_6
M_8	M_9	M_5
M_9	M_3, M_4, M_7, M_8	M_{10}
M_{10}	M_1, M_2, M_5, M_6	M_9
M_{11}	M_{13}, M_{14}	M_{13}
M_{12}	M_{13}, M_{14}	M_{14}
M_{13}	M_{11}, M_{12}	M_{11}
M_{14}	M_{11}, M_{12}	M_{12}

TABLE XIII

The point Z

Initial state	Final states	
	σ	π
Z_1	Z_3, Z_4	Z_1
Z_2	Z_3, Z_4	Z_2
Z_3	Z_1, Z_2	Z_4
Z_4	Z_1, Z_2	Z_3
Z_5	Z_5	Z_5

TABLE XIV

The point A

Initial state	Final states	
	σ	π
A_1	A_3, A_4	A_2
A_2	A_3, A_4	A_1
A_3	A_1, A_2	A_3
A_4	A_1, A_2	A_4
A_5	A_5	A_5

The author thanks Professor M. Suffczyński for suggesting the problem and for critical remarks.

INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT FIZYKI, PAN

REFERENCES

[1] K. Olbrychski, *Representations of the space group of rutile* (TiO_2), Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **7** (1961), 537.

[2] G. Lubarski, *The group theory and its applications in physics* [in Russian], Moscow, 1958, 248.

[3] J. S. Lomont, *Applications of finite groups*, New-York—London, 1959, 178.

[4] M. Sachs, *Selection rules for the absorption of polarized electromagnetic radiation by mobile electrons in crystals*, Phys. Rev., **107** (1957), 437.

[5] G. F. Koster, *Space groups and their representations*, Solid State Physics, **5** (1957), 173.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 12, 1961

EXPERIMENTAL PHYSICS

Raman Spectra of Pyridine Salts in Cristalline and Liquid States

by

R. MIERZECKI

Presented by A. JABŁOŃSKI on October 21, 1961

Introduction

Raman spectra of pyridine solutions have been the subject of frequent studies during the past thirty years. As an aromatic compound pyridine (C_5H_5N) shows bright Raman lines, and as the ring containing a nitrogen atom it is of basic character and has a noticeable dipole moment. For this reason strong intermolecular interactions occur between pyridine molecules and the molecules of the solvent. These interactions have a strong influence on the vibration spectra.

The most distinct changes occur in the symmetric vibrations of the ring (region 1000 K) and of C—H bond (region 3050 K). Some changes of antisymmetric vibrations (region 600 K) and of deformation vibrations of the ring (region 1600 K) are also observed.

The early investigations concerned the aqueous solutions of pyridine hydrochloride [1]—[5]. In such solutions two new frequencies at 1007 K and 1020 K were found in the Raman spectrum instead of two pyridine frequencies at 990 and 1029 K. These lines have been ascribed to the pyridine ions $C_5H_5NH^+$ present in the solutions studied. On the contrary, a new frequency at 1000 K was found in the spectra of solutions in which solvent molecules could form only a hydrogen bond with pyridine molecules (pyrrole, alcohol, water) [3], [4], [6]. This frequency has been ascribed to the vibrations of complexes formed by the pyridine with the solvent molecules.

In recent years various authors [4], [5], [7]—[10] investigated the spectra of solutions of pyridine in fatty acids. However, their results do not permit to draw unambiguous conclusions about the origin of the changes observed. In the spectra of solutions of certain acids only one frequency at 1003—1010 K was observed at some concentrations. In other cases a line at 1020—1025 K was also observed; as shown earlier by the author [10], the appearance of the latter depends on the presence of water in the investigated solutions.

These results indicate that the investigations of anhydrous pyridine salts may be of some interest. Under normal conditions the pyridine hydrochloride, hydro-

bromide and sulphate are solid, but their melting points are not high. The spectra of these salts in polycristalline and liquid states as well as in water solutions are compared.

Experimental

The pyridine hydrochloride has been prepared by saturating pyridine with gaseous hydrochloride using an arrangement shown in Fig. 1. A hydrochloric gas obtained from hydrochloric acid by a method of Sweeney [11] is introduced over a dehydrated pyridine distilled formerly into the vessel 5. A solid CO_2 trap 4 retains the traces of sulphuric acid that can be carried on by hydro-

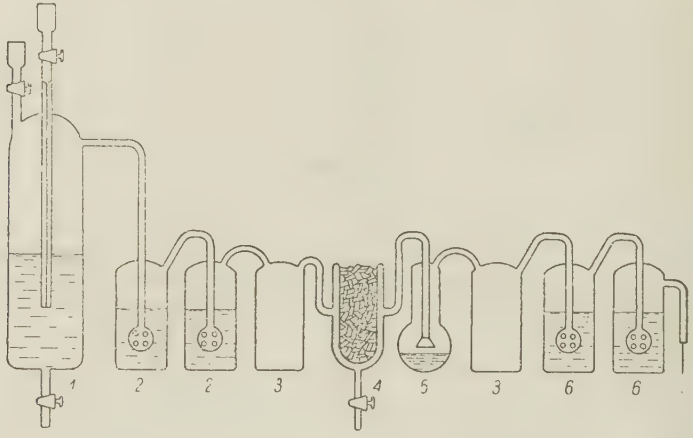

Fig. 1. Arrangement for pyridine hydrochloride

1. Dehydration vessel, 2. Washbottles with sulphuric acid. 3. Safety bottles, 4. Cold trap with solid CO_2. 5. Absorption vessel with pyridine. 6. Washbottles with kalium hydroxide solution

chloric gas. As the liquid in the vessel 5 is completely solidified, it is heated upon its melting point and poured into a Raman tube with a hemispherical ending. Lowering slowly the tube inside a resistance furnace with an appropriate temperature gradient a polycristalline specimen was obtained that was sufficiently translucent to enable discerning in its spectrum the strongest Raman lines against a strong background.

The pyridine hydrobromide produced by the firm Schuchardt München was used for similar investigations. The pyridine sulphate was prepared by distilling the dehydrated pyridine into the sulphuric acid of chemical purity.

The spectra were photographed by means of a Huett B II spectrograph with the prisms deviated from their normal position to enlarge the dispersion to 9 Å/mm. The exciting line was the Hg 4358 Å line (for the 3050 K region also the Hg 4047 Å line) from a low pressure mercury lamp of the Sosinski—Rezaev type [12]. The specimens were melted by means of a resistance heating wire wound on a glass tube in which the Raman tube was placed.

Results

The results of frequency measurements are given in Tables I, II and III. Some strong lines of pure pyridine are introduced for comparison. The most prominent differences between the spectra of po'ycristalline and liquid pyridine hydrochloride (Table I) appear in the symmetric vibrations of the ring. One finds only one frequency

at 1007 K in the spectra of polycristalline specimen, whereas in the spectrum of liquid specimen another frequency at 1020 K appears nearby. This is illustrated in Fig. 2 by photometric curves taken for a not totally hydrochlorated specimen in which the pyridine lines 990 K and 1029 K are also apparent.

Fig. 2. Photometer curves of spectrum of pyridine hydrochloride. The 990 K and 1029 K pyridine lines are also apparent because of an excess of pyridine in the specimen

Similar changes in the spectrum were also observed in the case of pyridine hydrobromide (Table II), however the line at 1020 K of liquid specimen is diffuse and forms a band together with the 1029 K line.

TABLE I

$C_5H_5N \cdot HCl$

C_5H_5N	Cristal.	Liquid	Aqueous solution concentrated	C_5H_5N +hydrochloric acid
601	606 vw	606 w	607 vw	608 vw
657	637 w	637 w	637 w	638 w
990				
	1007 s	1007 s	1008 s	1008 s
		1020 s	1024 m	1025 m
1029	1029 m	1029 w		
1048	1055 w	1055 w	1056 w	1059 w
1148				1161 vw
1216	1196 w	1191 m	1200 w	1202 m
	1255 vvw			1255 vw
1572				1616 vw
1580	1630 w	1630 vw	1632 w	1636 w
3056				
	3083 m	3086 s	3091 m	3098 m

vvw — very very weak, vw — very weak, w — weak, m — medium, s — strong, vs — very strong.

TABLE II

$C_5H_5N \cdot HBr$

C_5H_5N	Cristal.	Liquid	Aqueous solution concentrated
601	607 w	608 vw	608 w
	620 vw		
651	635 w	633 vw	638 m
990			995 w
	1008 vs	1007 s	1008 vs
		1018 w	1026 m
1029	1029 m	1029	
1048	1055 w	1049 w	1058 w
1145			1163 vw
1216	1188 w	1192 w	1200 s
	1240 vw		1254 vw
1572	1601 w	1612 vw	1614 w
1580	1632 w	1632 vw	1636 m
3055			
	3082 m	3083 m	3095 m

For explanations see Table I.

TABLE III

$C_5H_5N \cdot H_2SO_4$

With excess of H_2SO_4			With excess of C_5H_5N		
cristal.	liquid	aqueous solution	cristal.	liquid	aqueous solution
610 vw	610 vw	608 vw			
637 vw	637 vw	637 vw	638 vw	636 vw	636 m
				990 w	
1008 m	1010 s	1010 s	1008 w		1010 vs
			1020 m	1018 m	1020 m
1029 vw	1027 w	1028 m			
1041 vw		1047 w			
1058 vw	1062 w	1060 vw	1052 vw		1052 w
1199 w	1200 m	1202 w	1198 w		1204 m
		1251 vw			
		1639 vw			1634 vw
			3055 w		
3110 w	3107 m	3105 m	3095 m	3097 m	3100 m

For explanations see Table I.

In the case of pyridine sulphate there is a difference between the specimens prepared with an excess of pyridine and those prepared with an excess of acid. Nevertheless, no distinct changes of spectra were observed when specimens were melted or dissolved in water. Some changes in the spectra were, however, observed with the lapse of time.

Conclusion

As is seen from Table I, the frequencies at 1007 K and 1020 K ascribed to the pyridine ion, are discernible in the spectra of liquid specimens of anhydrous pyridine hydrochloride. Hence, one can conclude that pyridine ions exist also in the anhydrous salts.

There is only one frequency at 1007 K in the spectra of polycristalline specimens. Thus, one of the vibration modes of the ring is forbidden. This can result from interactions of pyridine ions with other ions or molecules. These interactions are very weak and disappear when specimens are melted; then the whole spectrum of ion can be seen.

The results of investigations of pyridine hydrobromide seem to confirm this conclusion.

For the spectra of pyridine sulphate no interpretation can be proposed, as there is a great possibility of sulphonation reaction and several chemical compounds can be present in the specimens under investigation.

The author is indebted to Dr. Skaliński for interesting suggestions in connection with this work.

INSTITUT OF EXPERIMENTAL PHYSICS, UNIVERSITY, WARSAW,
(INSTYTUT FIZYKI DOŚWIADCZALNEJ, UNIWERSYTET WARSZAWSKI)
ATOMIC AND MOLECULAR OPTICS LABORATORY, INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES
(ZAKŁAD OPTYKI ATOMOWEJ MOLEKULARNEJ, INSTYTUT FIZYKI, PAN)

REFERENCES

[1] P. Krishnamurti, Ind. J. Phys., **6** (1931), 367, 401.
[2] H. Bernstein, N. Martin, Trans. Roy. Soc. Canada, **3** (1937), 95.
[3] P. Bayard, Bull. Soc. Roy. Liege, **12** (1943), 179.
[4] P. Chiorboli, Ann. di Chim., **7** (1957), 443.
[5] G M. Schwab, G. Glatzer, Z.f. Elektrochem., **61** (1957), 1028.
[6] P. Chiorboli, P. Manares, Gazz. Chim. Ital., **84** (1954), 869.
[7] R. Mierzecki, Acta Phys. Polon., **12** (1953), 26.
[8] — , Bull. Acad. Polon. Sci., Cl. III, **3** (1955), 259.
[9] — , Current Science (India), (1956), 200.
[10] — , Acta Phys. Polon., **19** (1960), 41.
[11] O. R. Sweeney, J. Am. Chem. Soc., **39** (1917), 2186, (Inorganic Synthesis, NY, 1959).
[12] N. I. Rezaev, Viestnik MGU, (1957), 145.

TABLE DES MATIÈRES DU VOLUME IX

Mathématique

A. Suliński, A Classification of Semi-simple Rings 1

K. Maurin, Mappings of Hilbert-Schmidt Type. Their Application to Eigenfunction Expansions and Elliptic Boundary Problems 7

D. Zaremba, Sur la fixation et l'enfilage des ensembles compacts 13

A. Ehrenfeucht, Separable Theories 17

Chi Guan-Fu, A Note on Strong and Weak Convergence of Singular Integrals of Vector-Valued Functions 21

О. М. Фоменко, О сумме значений характера 29

M. Altman, An Iterative Method of Solving Functional Equations 57

M. Altman, Iterative Methods of Higher Order 63

W. Pogorzelski, Transformations des intégrales itérées singulières dans l'espace . . . 69

R. Engelking, On the Space of Measurable Sets of Real Numbers 75

W. Nitka, A Metrical Characterization of n-Cells 77

R. Bittner, On a New Definition of Polynomials 79

J. Meder, On the Estimation of (N, p_n)-Means of Orthogonal Series 123

J. Mioduszewski, On Two-to-One Continuous Functions 129

R. Bittner, Algebraic Properties of Linear Derivative Equations in Linear Spaces . . 133

R. Bittner, Algebraic Properties of Linear Derivative Equations in Algebras 141

T. Huskowski, Sur la normale à l'hypersurface de l'espace affine A_n 143

W. Roter, Sur l'application géodésique d'une variété riemannienne sur l'espace récurrent 147

T. Ważewski, Systèmes de commande et équations au contingent 151

W. Orlicz, A Note on Modular Spaces. I 157

A. Grzegorczyk, A. Mostowski, C. Ryll-Nardzewski, Definability of Sets in Models of Axiomatic Theories 163

M. Altman, A Linear Iterative Method with a Vector Parameter 169

M. Altman, An Iterative Method for Linear Algebraic Equations 175

M. Warmus, Transformed Point Arithmetic for Digital Computers 237

M. Warmus, Approximations and Inequalities in Calculus of Approximations. Classification of Approximate Numbers 241

Z. Frolík, On Almost Realcompact Spaces 247

H. Milicer-Grużewska, Propriété limite de la matrice du potentiel généralisé de simple couche du système parabolique d'équations 251

E. Sąsiada, Solution of the Problem of Existence of Simple Radical Ring 257

R. Sikorski, A Topological Characterization of Open Theories 259
M. Altman, Concerning Tchebyshev's Generalized Method of Solving Non-linear Functional Equations 261
M. Altman, Extension and Stability of Certain Iterative Methods for Solving Non-linear Functional Equations in Banach Spaces 267
S. Balcerzyk, On Classes of Abelian Groups 327
E. Sąsiada, Negative Solution of I. Kaplansky's First Test Problem for Abelian Groups and a Problem of K. Borsuk Concerning Cohomology Groups 331
L. Kublik, Some Remarks on the Class $\mathcal{L}$ of Probability Distributions 335
M. Król, Separable Groups. I 337
K. Borsuk, An AR-Set with an Infinite Number of $\mathcal{R}$-Neighbours 345
W. Pogorzelski, Propriétés d'une intégrale singulière dans l'espace 351
W. Pogorzelski, Problème linéaire aux dérivées tangentielles discontinues pour une fonction harmonique dans l'espace 357
K. Maurin, H_0^A-Räume und ihre Anwendung auf allgemeine Eigenfunktionsentwicklungen 363
A. Ehrenfeucht and A. Mostowski, A Compact Space of Models of First Order Theories 369
W. Kołodziej, Einige allgemeine Lückenumkehrsätze für permanente Toeplitzsche Limitierungsverfahren 375
R. Engelking, On the Freudenthal Compactification 379
R. Taberski, Summability with Korovkin's Factors 385
H. Milicer-Grużewska, Propriété limite du potentiel special de simple couche d'un système parabolique d'équations 429
A. Buraczewski, Determinant Systems for Generalized Fredholm Operators 435
R. Taberski, On Classes Λ_M^α and λ_M^α of 2π-Periodic Functions 441
W. Matuszewska, Some Further Properties of φ-Functions 445
S. Balcerzyk, A. Białynicki-Birula and J. Łoś, On Direct Decompositions of Complete Direct Sums of Groups of Rank 1 451
J. Radecki, On Modified Landau Polynomials 455
M. K. Fort, Jr., The Complements of Bounded, Open, Connected Subsets of Euclidean Space 457
H. Milicer-Grużewska, La quasi-solution spéciale du système parabolique d'équations 515
Dana Scott, Measurable Cardinals and Constructible Sets 521
W. Żakowski, Problème non linéaire et discontinu d'Hilbert pour le système de fonctions 525
P. Besals, On Solutions of Non-Linear Parabolic Equations Defined in Unbounded Domains 531
H. Milicer-Grużewska, La solution fondamentale spéciale du système parabolique d'équations 565
A. Wiweger, Some Applications of the Mixed Topology to the Theory of Two-norm Spaces 571
M. Altman, A Method of Steepest Ortho-descent 575
M. Altman, A Generalization of Laguerre's Method for Functional Equations . . . 581
S. Glidman, On Some Geometrical Properties of Normal Matrices 625
M. Altman, Concerning the Method of Tangent Hyperbolas for Operator Equations 633
M. Altman, An Iterative Method for Eigenvalues and Eigenvectors of Matrices . . . 639
M. Altman, A General Method of Steepest Ortho-descent 645
M. Altman, A Gradient Relaxation Method for Linear Algebraic Equations 653

R. Taberski, Some Properties of (K, φ)-summability 659
A. Lasota et Z. Opial, Sur un problème d'interpolation pour l'équation différentielle ordinaire d'ordre n 667
Z. Opial, Sur une équation différentielle presquepériodique sans solution presque-périodique 673
C. Bessaga, A. Pełczyński and S. Rolewicz, On Diametral Approximative Dimension and Linear Homogeneity of F-spaces 677
K. Borsuk, Concerning the Dimension of ANR-sets 685
D. Mangeron et L. E. Krivochéine, Sur quelques problèmes d'approximation relatifs à une nouvelle classe d'équations intégro-différentielles 707
W. Pogorzelski, Propriétés des intégrales itérées singulières dans l'espace 713
Z. Frolík, On Analytic Spaces 721
M. Altman, An Iterative Method for the Eigenproblem of Matrices 727
M. Altman, A Generalized Method of Steepest Descent 733
M. Altman, A General Method with Minimum Orthokesidua 739
M. Altman, A General Majorant Principle for Functional Equations 745
M. Altman, An Iterative Method for the Eigenproblem of Linear Operators 751
О. М. Фоменко, Оценки некоторых тригонометрических сумм 757
Z. Piesyk, The Existential and Universal Statements on Parallels 761
J. Musielak, On Some Spaces of Functions and Distributions. III. Spaces of Vector-valued M-integrable Distributions 765
R. Taberski, More about (K, φ)-summability 769
M. Altman, An Iterative Method for the Eigenproblem of Linear Bounded Operators 775
M. Altman, Two Iterative Methods for Linear Algebraic Equations 779
M. Altman, A General Deflation Principle for Linear Iterations 785
M. Altman, A Newton Type Method for Linear Equations 791
М. Бурнат, Устойчивость собственных функций и спектр для оператора Хилла . 795
Ian N. Sneddon, A Note on Some Relations between Fourier and Hankel Transforms 799
R. D. Anderson and E. H. Connell, Concerning Closures of Images of the Reals . . 807
D. Sadowska, Problème de Cauchy pour un système parabolique dont les coéfficients dépendent de fonctions inconnues 811
П. И. Петров, Классификация римановых пространств вполне вырожденной кривизны 817
K. Maurin, Distributionen auf Yamabe-Gruppen. Harmonische Analyse auf einer Abelschen l.k. Grupe 845
K. Maurin, Stability of the Index of Elliptic Boundary Value Problems 851
S. Moran, Note on a Question of Malcev 853
A. Pliś, Remark on Measurable Set-value Functions 857
T. Ważewski, Sur une condition d'existence des fonctions implicites mesurables . . . 861
T. Ważewski, Sur une condition équivalente à l'équation au contingent 865
T. Ważewski, Sur la sémicontinuité inférieure du „tendeur" d'un ensemble compact variant d'une façon continue 869
M. Altman, Concerning the Deflation Principle for Linear Iterations 873
M. Altman, Connection between Gradient Methods and Newton's Method for Functionals 877
R. O. Gandy, S. Kreisel and W. W. Tait, Set Existenie II 881

Astronomie

S. Gorgolewski, J. Hanasz, H. Iwaniszewski and Z. Turło, Log-Periodic-Aerial-Interferometer for Radioastronomy 689

Physique Théorique

J. Rayski, Conservation Laws in General Relativity 33
W. Woźnicki and S. Kwiatkowski, Estimation of the Influence of Foreign Atoms in the Core of Conjugated Bonds on the Spectrum and the Charge Distribution in the Molecule 39
W. Królikowski, On the Conservation of Isofermions and the Rule $\Delta S = \Delta Q$ in Weak Processes 43
R. Rączka, Evaluation of the Phase Space Integral in the Statistical Treatment of Multiple Production with Angular Momentum Conservation 93
L. Infeld, The EIH and the k-Approximation Methods 93
J. Czerwonko, On the Asymptotical Formula for the Thermodynamical Potential of Fermi Particles 99
A. Schild, On Gravitational Radiation Reaction Forces 103
W. Królikowski, Pionic Gauge 105
A. Witkowski, Electronic Spectra of Dimers. I. Nature of the Coupling 179
A. Witkowski, Electronic Spectra of Dimers. II. Perturbation Solutions of the Fundamental Vibronic Equation 185
A. Witkowski, Electronic Spectra of Dimers. III. Intensities 189
Jerzy Mycielski, On the Mechanism of Impurity Conduction in Semiconductors. I. The Activation Energy of Conductivity 193
Jerzy Mycielski, On the Mechanism of Impurity Conduction in Semiconductors. II. Conductivity in the Case of Strong Carrier-Phonon Interaction 199
J. Werle, On Some Problems of the S-Matrix Formalism for Unstable Particles . . . 207
Z. Koba, On the Angular Momentum Weight Factor in the Statistical Theory of Multiple Particle Production I 211
R. Żelazny and A. Kuszell, Milne's Problem For Two Adjacent Half Spaces 217
W. Ziętek, Quantentheoretische Ableitung der magnetischen Struktur anisotroper ferromagnetischer Einkristalle 221
J. Mozrzymas and J. Rzewuski, Second Order Invariant Differential Equation in the Spinor Space 225
W. Woźnicki, Excited States of Acetylene 273
R. Gałązka and W. Giriat, Electrical Properties of the CdTe-HgTe System 281
W. Hanus, On Relative Doublet Line Strengths in the Principal Series of Cesium . . . 287
J. Kołodziejczak, Electrons and Holes in Indium Antimonide 293
L. Infeld, On the Most Cartesian-like Co-ordinate System 299
W. Królikowski, A Possible Mechanism of the $D_{3\,2}$ $T = 1/2$ Resonance in Pion-Nucleon Scattering 303
A. Krzywicki, On the Final-State Interaction in Single-Pion Production Processes . . . 305
R. S. Ingarden and K. Urbanik, Information as a Fundamental Notion of Statistical Physics 313
J. Lukierski, On the Antilinearization of the Klein-Gordon Equation for 4-Spinor . . . 317
B. Mielnik, On the Damping Problem in Quantum Theory 389

Ziro Koba, On the Angular Momentum Weight Factor in the Statistical Theory of Multiple Particle Production II 395
R. Żelazny and A. Kuszell, A Special Model of a Two-Group Approach in Neutron Transport Theory 461
W. Garczyński, On Some Representation of Perturbation Expansion of Scattering Amplitude 467
W. Garczyński, Some Topological Properties of Feynman Diagrams 473
M. Miąsek and M. Suffczyński, Space Group of White Tin. I. Symmetry Points . . 477
M. Miąsek and M. Suffczyński, Space Group of White Tin. II. Symmetry Lines and Planes 483
M. Suffczyński, Space Group of White Tin. III. Double Group 489
K. Olbrychski, Representations of the Space Group of Rutile (TiO_2) 537
F. A. E. Pirani and A. Schild, Geometrical and Physical Interpretation of the Weyl Conformal Curvature Tensor 543
R. S. Ingarden, Quantum Phenomena in Optical Images 549
A. Jaskiewicz and H. Konwent, Dipole Array of Ferroelectrically Active *A*-Ions in ABO_3-Substances 553
J. Fiutak and L. Wolniewicz, Vibrational Excitation of Diatomic Molecules Due to Nuclear Recoil 557
J. Plebański, On Algebraical Properties of Skew Tensors 587
J. Plebański, On Some Applications of Algebraical Properties of Skew Tensors . . . 593
W. Ziętek, Die magnetischen Strukturen im hexagonalen Gitter 599
Z. Galasiewicz, On the Wave Function of the Anomalous State of Fermi-system . . . 605
M. Miąsek and M. Suffczyński, Space Group of White Tin. IV. Basic Functions for the Irreducible Representations at Symmetry Points 609
L. Infeld, Is Planck's Constant a Constant in Gravitational Field? 617
M. Demiański and E. Infeld, Note on the Field Method of Obtaining the Conservation Laws and Solving the Two Body Problem in General Relativity 693
A. Witkowski, On Transitions Involving Crossing of Potential Energy Curves . . . 697
W. Garczyński, On the Majorization Problem in Perturbation Theory 819
J. Pachner, Allgemeine Relativitätstheorie zur Zeitdilatation bei beschleunigter geradliniger Bewegung 827
A. Wierzbicki, Atomic Theory of Light Propagation, Refraction and Reflection in Isotropic Media, with Consideration of Electric Quadrupole Radiation 833
K. Olbrychski, Selection Rules for Electric Dipole Transitions in Rutile 883

Physique Expérimentale

B. Bączyński, M. Czajkowski and S. Trawiński, On the Fluorescence of Fluorescein in Methyl Methacrylate 47
D. Frąckowiak and T. Marszałek, Yield of Anti-Stokes Fluorescence of Chlorophyl 53
L. Czuchajowski, M. Lasoń, W. Szymanowski, A. Kujawski, I. Olszewskal A. Góralczyk, Infrared Absorption Spectra of Polish Coals by Pressed Powder Metho,d 107
J. Głowacki, The Influence of Tartaric Acid on the Photoluminescence of Dye Solutions 113
K. Małuszyńska, I. Natanson and Yuan-Han-Yung, Azimuthal Angular Distribution of Particles from the $^{10}B(n, \alpha)^7$ Li Reaction with Polarized Thermal Neutrons 119
B. Hilczer, Effect of γ ^{60}Co Radiation on the Dielectric Properties of Triglycine Sulphate 229

H. Łożykowski and H. Męczyńska, An Attempt to Obtain Electroluminescence of Organophosphors 235
B. Moszyńska, Variation de l'intensité des raies Raman avec la température 323
E. Igras, Mapping of the Electric Structure of Semiconductor Surfaces by the Method of the Electron Mirror 403
L. Kozłowski and S. Kubiak, Effect of Cathodic Hydrogen on the Magnetic Properties of Thin Electrolytic Nickel Films 409
J. Ginter and W. Szymański, Measurement of Thermoelectric Power in InSb 419
J. Heldt, Irreversible Photobleaching of Acriflavine in Polyvinyl Alcohol 497
A. Gutsze and H. Walery$_0$, Influence of the Concentration of the Solute and of the Polarity of the Solvent on the Absorption Spectra of Fluorescein 507
S. Czarnecki, Anomalous Phosphorescence of Naphthalene in Methyl Methacrylate Polymer 561
K. Małuszyńska, L. Natanson, J. Turkiewicz and P. Żuprański, Angular Distributions of Fast Neutrons Elastically Scattered on Ca 621
R. Drabent and A. Jachym, Spectral Investigation of Rhodamine Polymerization Processes in Collodion 701
A. Gutsze and H. Walery$_0$, Absorption Spectra of Eosin in Media of Various Polarity and *pH* 839
R. Mierzecki, Raman Spectra of Pyridine Salts in Cristalline and Liquid States . . . 889

Géophysique

R. Teisseyre, A Dislocation Theory of the Earthquake Processes 423

LISTE DES AUTEURS DU VOLUME IX

Altman M. 57, 63, 169, 175, 261, 267, 575, 581, 633, 639, 645, 653, 727, 733, 739, 745, 751, 775, 779, 785, 791, 873, 877
Anderson R. D. 807
Balcerzyk S. 327, 451
Bączyński B. 47
Bessaga C. 677
Besala P. 531
Białynicki-Birula A. 451
Bittner R. 79, 133, 141
Borsuk K. 345, 685
Buraczewski A. 435
Бурнат M. 666, 795
Chi Guan-Fu 21
Connell E. H. 807
Czajkowski M. 47
Czarnecki S. 561
Czerwonko J. 99
Czuchajowski L. 107
Demiański M. 693
Drabent R. 701
Ehrenfeucht A. 17, 369
Engelking R. 75, 379
Fiutak J. 557
Фоменко O. M. 29, 757
Fort M. K. 457
Frąckowiak D. 53
Frolík Z. 247, 721
Galasiewicz Z. 605
Gałązka R. 281
Gandy R. O. 881
Garczyński W. 467, 473, 819
Ginter J. 419
Giriat W. 281
Glidman S. 625
Głowacki J. 113
Gorgolewski S. 689
Góralczyk A. 107
Grzegorczyk A. 163
Gutsze A. 507, 839
Hanasz J. 689
Hanus W. 287
Heldt J. 497
Hilczer B. 229
Huskowski T. 143
Igras E. 403
Infeld E. 693
Infeld L. 93, 299, 617
Ingarden R. S. 313, 549
Iwaniszewski H. 689
Jachym A. 701
Jaśkiewicz A. 553
Koba Z. 211, 395
Kołodziej W. 375
Kołodziejczak J. 293
Konwent H. 553
Kozłowski L. 409
Kreisel G. 881
Krívochéine L. E. 707
Król M. 337
Królikowski W. 43, 105, 303
Krzywicki A. 305
Kubiak S. 409
Kublik L. 335
Kujawski A. 107
Kuszell A. 217, 461
Kwiatkowski S. 39
Lasoń M. 107
Lasota A. 667
Lukierski J. 317
Łoś J. 451
Łożykowski H. 235
Małuszyńska K. 119, 621
Mangeron D. 707
Marszałek T. 53
Matuszewska W. 445
Maurin K. 7, 363, 845, 851
Meder J. 123
Męczyńska H. 235
Miąsek M. 477, 483, 609

Mielnik B. 389
Mierzecki R. 889
Milicer-Grużewska H. 251, 429, 515, 565
Mioduszewski J. 129
Moran S. 853
Mostowski A. 163, 369
Moszyńska B. 323
Mozrzymas J. 225
Musielak J. 765
Mycielski Jan 873
Mycielski Jerzy 193, 199

Natanson L. 119, 621
Nitka W. 77

Olbrychski K. 537, 883
Olszewska I. 107
Opial Z. 667, 673
Orlicz W. 157

Pachner J. 827
Pełczyński A. 677
Петров П. И. 666, 817
Piesyk Z. 761
Pirani F. A. E. 543
Plebański J. 587, 593
Pliś A. 857
Pogorzelski W. 69, 351, 357, 713

Radecki J. 455
Rayski J. 33
Rączka R. 85
Rolewicz S. 677
Roter W. 147
Ryll-Nardzewski C. 163
Rzewuski J. 225

Sadowska D. 811
Sąsiada E. 257, 331
Schild A. 103, 543
Scott Dana 521
Sikorski R. 259
Sneddon Ian N. 799
Steinhaus H. 873
Suffczyński M. 477, 483, 489, 609
Suliński A. 1
Szymanowski W. 107
Szymańska W. 419
Taberski R. 385, 441, 659, 769
Tait W. W. 881
Teisseyre R. 423
Trawiński S. 47
Turkiewicz J. 621
Turło Z. 689
Urbanik K. 313
Waleryś H. 507, 839
Warmus M. 237, 241
Ważewski T. 151, 861, 865, 869
Werle J. 207
Wierzbicki A. 833
Witkowski A. 179, 185, 189, 697
Wiweger A. 571
Wolniewicz L. 557
Woźnicki W. 39, 273
Yuan-Han-Yung 119
Zaremba D. 13
Ziętek W. 221, 599
Żakowski W. 525
Żelazny R. 217, 461
Żuprański P. 621

БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК
СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ
И ФИЗИЧЕСКИХ НАУК
1961
ТОМ IX Резюме статей ВЫПУСК 12

Х. МОРЭН, **ОБОБЩЕННЫЕ ФУНКЦИИ НА ГРУППАХ ЯМАБЭ. ГАРМОНИЧЕСКИЙ АНАЛИЗ НА КОММУТАТИВНЫХ ГРУППАХ** . . . . . .стр. 845—850

Теорема 1. *Пусть G — (сепарабельная) локально компактная группа Ямабэ или дискретная группа. Тогда имеется функциональное пространство $\Phi \subset C_0(G)$, обладающее следующими свойствами:* 1° *Φ — ядерное,* 2° *Φ — равномерно плотное в $C_0(G)$,* 3° *Φ — плотное в $L^p(G)$,* 4° *левые операторы сдвигов U_S индуцируют сильно непрерывное представление в пространстве Φ,* 5° *операторы U_S непрерывны в Φ.*

Из Теоремы 1 и общей спектральной теоремы автора легко вытекают основные теоремы гармонического анализа на коммутативных группах (Вейля-Райкова-Крейна), а также далеко идущие обобщения известной теоремы Шварца-Бохнера о положительно определенных обобщенных функциях.

Х. МОРЕН, **СТАБИЛЬНОСТЬ ИНДЕКСА ЭЛЛИПТИЧЕСКИХ КРАЕВЫХ ЗАДАЧ** стр. 851—852

В работе доказывается, что индекс эллиптических краевых проблем не подвергается изменениям при возмущениях низшего порядка. Метод, примененный автором в работе, опирается на то обстоятельство, что утверждения Браудера и Кондрашева дают возможность использовать абстрактные утверждения о стабильности индекса замкнутого оператора в пространстве Банаха.

С. МОРАН, **ЗАМЕТКА О ПРОБЛЕМЕ МАЛЬЦЕВА** . . . . . .стр. 853—855

В работе дается пример регулярного произведения групп, коммутативного и ассоциативного, обладающего притом таким свойством, что гомоморфизмы отдельных множителей регулярного произведения вытягиваются исключительно для получения гомоморфизма регулярного произведения. Упомянутое произведение удовлетворяет следующим условиям:

(iii)′ если регулярное произведение не равно свободному произведению одинаковых множителей, то оно не может быть разложено на свободное произведение;

(iv)′ если в регулярном произведении выделить „серийные" подгруппы из каждого регулярного множителя, то эти подгруппы порождают группу изоморфную такому-же регулярному произведению этих подгрупп.

(Здесь как „серийные" понимаются подгруппы, обладающие нормальными вполне упорядоченными системами, ведущими от подгруппы до группы).

А. ПЛИСЬ, **ЗАМЕТКА О ИЗМЕРИМЫХ ФУНКЦИЯХ МНОЖЕСТВ** стр. 857—859

Основанием работы является Теорема 1.

Теорема 1. *Пусть $F(x)$ — функция определенная на кубе A со значениями из семейства непустых замкнутых подмножеств некоторого компактного метрического пространства S. Полагается, что множество*

(1) $(x : F(x) \subset Z)$ *измеримо для каждого замкнутого множества $Z \subset S$.*

Опираясь на упомянутые предположения, получаем следующее утверждение: для произвольного положительного числа r имеется такое подмножество B множества A, что функция $F(x)$ непрерывна на B, причем в пространстве множеств $Z \subset S$ мы пользуемся расстоянием Гаусдорффа и мера $A - B$ меньше r.

Предположение (1) справедливо, в особенности, для функции $F(x)$ полунепрерывных снизу (по отношению к включению).

Т. ВАЖЕВСКИЙ, **О НЕКОТОРОМ УСЛОВИИ СУЩЕСТВОВАНИЯ НЕЯВНЫХ ИЗМЕРИМЫХ ФУНКЦИЙ** . . . . . стр. 861—863

В Теореме 2 дается некоторое условие достаточное для того, чтобы существовала одна, по крайней мере, функция, неявная и измеримая, удовлетворяющая некоторым соотношениям, полагая, что нам известно, что упомянутым соотношениям удовлетворяют некоторые неявные функции, необязательно измеримые.

Теорема 2 связана с проблемой — рассматриваемой автором в предыдущих работах — эквивалентности управляемой системы дифференциальных уравнений соответствующему контингенсному уравнению. Теорема 2 следует из Теоремы 1, которая приведена вместе с доказательством.

Автор обращает внимание на то обстоятельство, что Теорема 1 может быть доказана и другим путем, опираясь на идею, примененную автором в одной из предыдущих работ.

Т. ВАЖЕВСКИЙ, **НЕКОТОРОЕ УСЛОВИЕ РАВНОСИЛЬНОЕ КОНТИНГЕНСНОМУ УРАВНЕНИЮ** . . . . . стр. 865—867

Исходя из контингенсного уравнения K (в смысле А. Маршо и С. К. Зарембы) автор дает некоторое условие W, равносильное уравнению K, причем в условии W не применяется понятие контингенсной производной.

Т. ВАЖЕВСКИЙ, **О НИЖНЕЙ ПОЛУ-НЕПРЕРЫВНОСТИ „ТЕНДОРА" КОМПАКТНОГО МНОЖЕСТВА ИЗМЕНЯЮЩЕГОСЯ НЕПРЕРЫВНО** . . . стр. 869—872

Тендор компактного множества A — это его наименьшая компактная часть, обладающая таким-же выпуклым прикрытием, как и множество A.

В работе дается следующая теорема: *Если компактное множество $A = A(t)$ является непрерывной функцией времени t (в смысле Гаусдорффа), то тендор множества $A(t)$ является нижне полу-непрерывной функцией (принимая во внимание включение).*

М. АЛЬТМАН, **ПО ПОВОДУ ПРИНЦИПА ДЕФЛЯЦИИ ДЛЯ ЛИНЕЙНЫХ ИТЕРАЦИЙ** . . . . . стр. 873—876

В работе дается другой принцип дефляции для линейных итерационных процессов.

Сущность этого принципа состоит в сведении итерационного метода к подпространству. Этим путем улучшается плохая обусловленность линейного оператора, что в результате дает ускорение сходимости метода. Работа содержит также применение к решению систем линейных алгебраических уравнений.

М. АЛЬТМАН, **СВЯЗЬ МЕЖДУ ГРАДИЕНТНЫМИ МЕТОДАМИ И МЕТОДОМ НЬЮТОНА ДЛЯ ФУНКЦИОНАЛОВ** . . . . . стр. 877—880

В работе доказывается, что известные градиентные методы являются частными случаями обобщения метода Ньютона для нелинейных функционалов в пространстве Банаха. Таким образом, получаются различные методы градиентов в зависимости от выбора нелинейного функционала и соответственного пространства.

К. ОЛЬБРЫХСКИЙ, **ПРАВИЛА ОТБОРА ДЛЯ ЭЛЕКТРИЧЕСКИХ ДИПОЛЬНЫХ ПЕРЕХОДОВ В РУТИЛЕ** . . . . . стр. 883—888

В первой части работы дается — в сокращенном виде — общая теория правил отбора для электрического дипольного излучения в ее применении к пространственным группам. Учитывается притом спин-орбитальное взаимодействие а также показана зависимость правил отбора от направления поляризации света.

Вторая часть сообщения — это перечень правил отбора для рутила. Приведена возможность оптических переходов из состояний с волновыми функциями, преобразующимися по неприводимым представлениям двойных групп $G(k)$ волнового вектора k, причем k пробегает все характеристические точки, оси и плоскости зоны Бриллюэна.

Р. МЕЖЕЦКИЙ, **СПЕКТРЫ РАМАНА ПИРИДИНОВЫХ СОЛЕЙ В КРИСТАЛЛИЧЕСКОМ И ЖИДКОМ СОСТОЯНИЯХ** . . . . . .стр. 889—893

С целью выяснить происхождение линии 1020 К, появляющейся в спектрах Рамана водяных растворов хлоргидрата пиридина, а также некоторых растворов пиридина в жирных кислотах, были сопоставлены спектры ангидридных солей пиридина в кристаллическом и жидком состояниях.

Констатировано, что упомянутая линия появляется в спектрах жидких солей, но отсутствует в спектрах кристаллических солей.

Эта разница обусловлена междумолекулярным воздействием.

TABLE DES MATIÈRES

Mathématique

1. K. Maurin, Distributionen auf Yamabe-Gruppen. Harmonische Analyse auf einer Abelschen l.k. Gruppe 845
2. K. Maurin, Stability of the Index of Elliptic Boundary Value Problems 851
3. S. Moran, Note on a Question of Malcev 853
4. A. Pliś, Remark on Measurable Set-valued Functions 857
5. T. Ważewski, Sur une condition d'existence des fonction implicites mesurables . . 861
6. T. Ważewski, Sur une condition équivalente à l'équation au contingent 865
7. T. Ważewski, Sur la sémicontinuité inférieure du „tendeur" d'un ensemble compact variant d'une façon continue 869
8. M. Altman, Concerning the Deflation Principle for Linear Iterations 873
9. M. Altman, Connection between Gradient Methods and Newton's Method for Functionals 877
10. R. O. Gandy, G. Kreisel and W. W. Tait, Set Existence II 881

Physique Théorique

11. K. Olbrychski, Selection Rules for Electric Dipole Transitions in Rutile 883

Physique Expérimentale

12. R. Mierzecki, Raman Spectra of Pyridine Salts in Cristalline and Liquid States . . 889

Table des matières du volume IX 895
Liste des auteurs 901

Cena zł 20.—